SURVEY AND DEVELOPMENT OF FINITE ELEMENTS FOR NONLINEAR STRUCTURAL ANALYSIS

VOLUME I – HANDBOOK FOR NONLINEAR FINITE ELEMENTS

FINAL REPORT

NAS 8-30626

(NASA-CR-144276) SURVEY AND DEVELOPMENT OF FINITE ELEMENTS FOR NONLINEAR STRUCTURAL ANALYSIS. VOLUME 1: HANDBOOK FOR NONLINEAR FINITE ELEMENTS Final Report (Boeing Aerospace Co., Seattle, Wash.) 165 p HC G3/39

N76-22582
HC $6.75

Unclas
26833

MARCH 16, 1976

Prepared by

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Prepared For

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
GEORGE C. MARSHALL SPACE FLIGHT CENTER
MARSHALL SPACE FLIGHT CENTER, ALABAMA 35812

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TABLE OF CONTENTS

1.0 INTRODUCTION AND PURPOSE

This document presents a comprehensive survey of past and current research efforts in the area of geometrically nonlinear finite elements. The survey is intended to serve as a guide in the choice of nonlinear elements for specific problems, and as background to provide directions for new element developments. The elements are presented in a handbook format and are separated by type as beams, plates (or shallow shells), shells, and other elements. Within a given type, the elements are identified by the assumed displacement shapes and the forms of the nonlinear strain equations. Solution procedures are not discussed except when a particular element formulation poses special problems or capabilities in this regard.

The main goal of the format is to provide quick access to a wide variety of element types, in a consistent presentation format, to facilitate comparison and evaluation of different elements with regard to features, probable accuracy, and complexity. References are provided to aid in further detailed studies which readers may wish to undertake.

The survey is based primarily on published literature. In addition, however, in order to have an up-to-date list, researchers have been contacted by letter and requested to prepare inputs to this document based on their most current efforts.

2.0 SCOPE

The available literature on nonlinear finite elements is quite large. The types of elements, the basis of derivations and solutions, and the types of nonlinearities considered are all extensively documented in the literature. Some limitations have been imposed on this document in order to obtain a useful yet comprehensive compilation of nonlinear elements.

First, emphasis has been placed on geometric nonlinearities as opposed to material nonlinearities. Geometric nonlinearities involve basic assumptions concerning strain-displacement equations and the development of new elemental matrices. They also strongly influence the relative adequacy of element displacement functions. Thus, the geometric nonlinearities of the formulations are intrinsic to the elements, while extensions to nonlinear material characterization can be incorporated in most elements by employing numerical integration schemes in calculations of element properties, and by using highly competent solution procedures. For these reasons, the research for this document has been directed toward geometric nonlinearities and material nonlinearities have only been included as an adjunct to this goal.

Secondly, only finite elements based on the displacement method have been considered. Neither the force or equilibrium elements, nor the various hybrid elements have been included. This restriction is motivated by the fact the hybrid and force method elements employ a considerably different approach than displacement elements, and are difficult to classify in the latter group.

Third, solution methods have not been emphasized in the document. This subject has been reviewed rather extensively in the published literature. Occasionally, the derivation of elemental matrices is tied closely to the solution procedure used to solve the structural problem. In these cases the usefulness or accuracy of the element may not be immediately evident by simply considering elemental matrices. However, in general, a variety of solution procedures can be used with any given element formulation. Thus, the omission of this subject herein does not compromise the usefulness of the element descriptions given. In

general, a variety of solution procedures can be used with any given element formulation. Thus, the omission of this subject herein does not compromise the usefulness of the element descriptions given. In some cases where solution procedures are specialized for a certain element, they are discussed briefly.

Qualitative comparison of the various elements has not been made. Those advantages and disadvantages and relative accuracy data that have been noted in the literature are repeated where the data appear particularly important.

Finally, the element descriptions in the handbook include elements for which complete formulations and examples of problem solutions have been published. There are a number of very general finite element formulations which have not been specialized to apply to particular elements, but which could be used to generate a number of related elements. Such general formulations are described in References 104 through 107. The last reference is particularly complete in this regard.

3.0 USER'S GUIDE

The handbook is composed of four parts based on element type. These basic types are:

B-1,3,----	Beam or line elements
P-1,2,----	Plate or shallow shell elements
S-1,2,----	Shell elements
C-1,2,----	Solid or three-dimensional elements

Within a given element type, the elements are catalogued by displacement function. Each new displacement function is given a new element I.D. (i.e., P-2, P-3, P-4). When several authors have used the same displacement function for a given element but differences in scope, features, and capabilities exist between the resulting elements, then the element identification retains the basic I.D., but carries a lower case alphabetic delineator (i.e., B-2a, B-2b, B-2c). The handbook description given for each element includes the I.D. number and a brief definition of element type, followed by the displacement function equations and concise description of the element in narrative form. This is followed by the reference where the element was derived, variations from the basic element, and any known advantages or disadvantages of the element. The strain displacement equations are given to indicate how the geometric nonlinearities are introduced into the strains, and what nonlinearities are included in the formulation. This is followed by a discussion of the element in which user-orientated information concerning material description, coordinate systems, and solution procedures are presented.

No attempt has been made to list chronologically the authors of various finite element concepts. In many cases several authors have published slight variations of a single element, and it appears preferable to deal with the elements themselves rather than the chronology of development. The order of the references given in this document is thus quite random.

In many cases the plate elements are applicable to shallow shells. The classification of such elements overlaps to some extent. It is therefore suggested that the reader refer to both the section on plates and the section on shells for each application.

4.0 NOMENCLATURE

In forming a reference book of this kind, based solely on the published works of many authors, a great variety of nomenclatures were encountered. Early in this project the decision was made not to convert the variety of nomenclatures to a common nomenclature (a formidable task) but rather to use the reference authors own nomenclature in the writeup for that reference. The reason being that the purpose of this book is not to present programmable matrices but rather to gather together the published nonlinear elements and provide sufficient information for a reader to direct his attention to a particular reference of interest. The reference material is obviously the best source for a detailed study of the elemental matrices, elemental characteristic, solution procedure, and applications.

Much of the nomenclature is common in the various references, such as:

E	-	Young's Modulus
σ	-	stress
ε	-	strain
[K]	-	stiffness matrix
$[K^\circ]$	-	linear stiffness matrix
$[K']$	-	K - one geometric stiffness matrix
$\{\delta\}$	-	column matrix of nodal displacements
u,v,w	-	displacements
x,y,z	-	rectangular cartesian coordinates
DOF	-	degrees of freedom

When an author uses unique symbols or nonstandard nomenclature these terms are defined and in addition, one or more figures are generally used to illustrate the authors particular notation. Subscripts are often used to denote derivatives. Their use is obvious in most applications.

5.0 NONLINEAR STRAINS, COORDINATE SYSTEMS, AND SOLUTION PROCEDURES

This section briefly discusses the sequence of calculations, particularly those associated with stress and stiffness matrix calculation, which is involved in nonlinear finite element problem solutions. The goal of this discussion is to offer some perspective concerning the several ways of handling the nonlinear strain displacement equations. The subjects discussed here are a part of the overall subject of solution procedures for nonlinear finite element problems. Several excellent references are available which discuss this subject in more detail and from different viewpoints from those used here. References 104 through 107 are particularly helpful in this regard.

Global Coordinate System and Nonlinear Strain Equations

Finite element derivations nearly always adhere to the basic Lagrangian formulation of mechanics. In this formulation the displacements and forces are referred to a fixed, global coordinate system. The strains and stresses are "convected" with the material as the deformation progresses and strains are computed by referring distortions to the initial lengths and orientations of the material elements. For a cartesian global coordinate system, the strain of the material "fiber" initially aligned with the x axis is given by the familiar formula, which occurs frequently in the element descriptions in this report,

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right]$$

The incremental strain is given by,

$$\Delta\varepsilon_x = \frac{\partial \Delta u}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial \Delta u}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial \Delta v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial \Delta w}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial \Delta u}{\partial x}\right)^2 + \left(\frac{\partial \Delta v}{\partial x}\right)^2 + \left(\frac{\partial \Delta w}{\partial x}\right)^2\right]$$

ε_x is actually a component of the so-called second Piola-Kirchhoff strain tensor, rather than a true physical strain. However, for small strains, for cartesian coordinate systems, it can be taken as the physical strain with sufficient accuracy.

Figure 1 illustrates for a very simple case two facts concerning the application of the above equations which are particularly important for finite element calculations. In the figure, the w and u displacements are both large and $(\frac{\partial u}{\partial x})^2$ and $(\frac{\partial w}{\partial x})^2$ are of roughly equal importance in the calculation of the strain even though the strain is small. Note that $\frac{\partial u}{\partial x}$ is not a good approximation to the strain. In the early stages of the movement from the initial to the final positions of the bar element, however, $\frac{\partial u}{\partial x}$ is small and is a good approximation to the strain. Hence, early in the deformation, i.e., while the bar is still close to parallel to the x axis, the term $(\frac{\partial u}{\partial x})^2$ can be omitted in the strain equation, though $(\frac{\partial w}{\partial x})^2$ must be retained.

Figure 1 also illustrates the fact that physical meanings of the u and w displacements change as the bar rotates away from the position parallel to the x axis. Early in the deformation, u is essentially an axial displacement, and w is a lateral, or bending, type of displacement. This is significant for finite element work, because usually the element displacement functions used for the axial and bending types of displacements differ from one another, with the bending displacement functions being considerably more competent than the axial functions. In the latter stages of the deformation illustrated by the figure, the u displacement has the nature of a bending displacement. If simple element functions were used for u, they would thus become incompetent for large deformations.

If the global coordinate system is curvilinear, as it might be for a shell problem, the nonlinear strain equations become more complicated. The essential features of the above discussion remain unchanged, however.

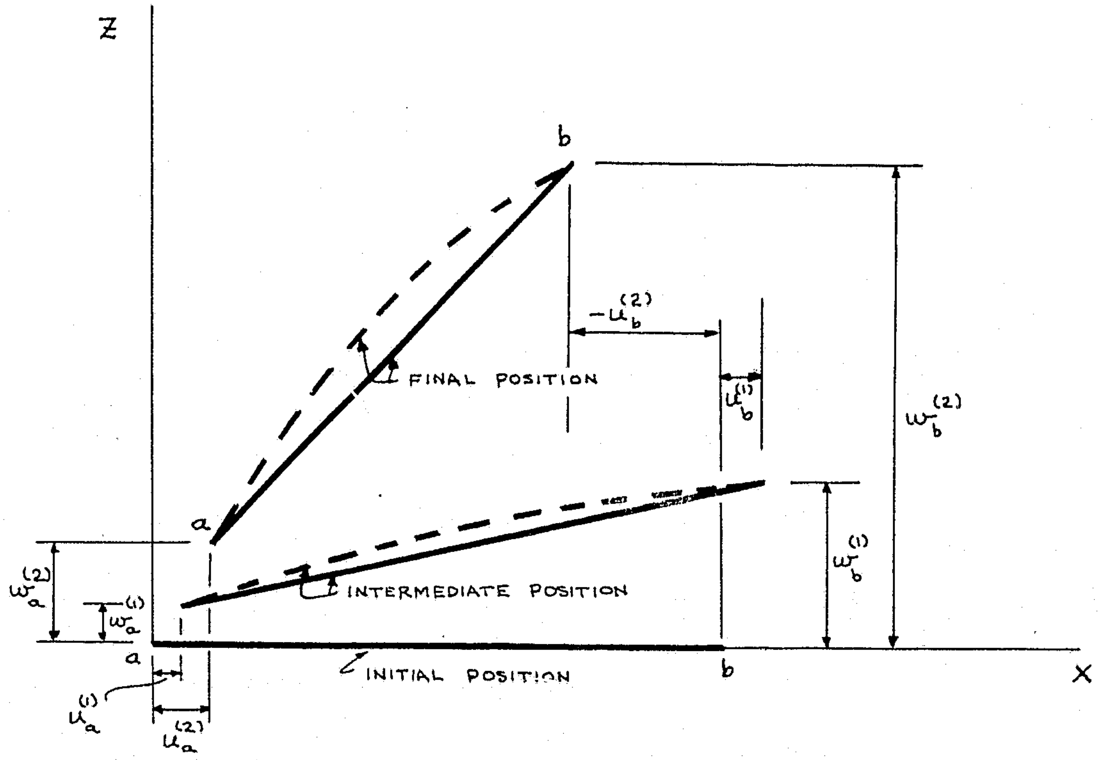


FIGURE 1 : ILLUSTRATION OF LARGE DEFLECTION / SMALL STRAIN STATE FOR STRAIGHT AND BENT BAR

A final point needs to be made concerning the nonlinear strain equation. Figure 1 shows both solid-line and dashed-line positions of the bar, corresponding to non-bending and bending types of problems, respectively. The approximation is frequently made, for bending problems, that the nonlinear terms in the strain equation only include the straight-line element shape, i.e., ignore the bending curvature. Thus, the use of the term $\left(\frac{\partial w}{\partial x} \right)^2$ in an element formulation is not completely definitive; the w may or may not include the element "bowing" during the actual calculation of the strain.

Stiffness Matrices, Incremental Solutions

If the nonlinear strain equations are substituted into an energy or virtual work integral to derive the element incremental equilibrium equations, a number of different linear and nonlinear terms are obtained. Defining Δq, K, and ΔP, respectively, to be element incremental displacements, stiffness matrices, and incremental loads, the linearized matrix incremental quilibrium equation has the form

$$(K_o + K_G)\Delta q = \Delta P$$

The Ko matrix contains the stiffness of the element against very small deformations caused by the incremental displacements. K_o is not a constant matrix, since it includes the effects of accumulated deformations prior to the increment in question. K_o can be thought of as containing two parts, the first representing linear behavior associated with very small displacements from the initial, undeformed state; and the second accounting for the changes in stiffness due to the accumulated rotations. Often only the first part is used in an element formulation, resulting in a constant K_o matrix. K_G contains stresses, and provides an "effective" stiffness effect associated with the changes of direction of the stresses in the element as the rotations due to Δq take place. The K_G matrix, often called the geometric, differential, or initial stress stiffness matrix, arises from the nonlinear terms in the strain equation. The non-constant part of K_o arises from these same terms, and the linearization to obtain a constant K_o matrix therefore

results from dropping the same nonlinear terms which gave the K_G matrix. The above linearized incremental equilibrium equation has itself resulted from dropping second order terms in Δq, these terms also arising from nonlinear terms in the strain equations.

It is seen that an element's use of nonlinear strain equations, such as those given in the element descriptions in this handbook, does not necessarily mean that the nonlinear element in question will in calculations demonstrate fully nonlinear behavior. The nonlinear character of the element depends on how the nonlinear strains equations are used in the element derivation and in problem solutions. Several options are used in this regard, partly associated with solution procedures, and partly with the element equations themselves.

The NASTRAN program uses a simple but often effective procedure in which no nonlinear strain development is included at all, but the K_G effect itself is retained. In this case the strain and stress development is based on the linearized strain-displacement equations, the K_O matrix is the constant initial state matrix, and through the K_G matrix the nonlinear effects of the rotations on the equilibrium equations are included. NASTRAN actually uses in K_G the stresses at the end of the load step, requiring iteration to obtain this solution. The solution obtained is a correct nonlinear solution provided only that the nonlinear strain contributions to the final stress state are negligible. The presence of a dominant initial stress state is not needed, though it would increase accuracy with this sort of approach. A variant of the NASTRAN procedure is a stepwise calculation in which the stresses at the start of each increment are used in K_G. This would eliminate or reduce the need for iteration, at the expense of accumulating error.

Another simple approach which has found widespread use in nonlinear finite element work also works without reference to any nonlinear strain description. In the derivation of the equations of a particular finite element, frequently the geometry of the element is described in a fairly general way from which simpler geometries are available by specialization. For

example, a general quadrilaterial element might be considered, and its stiffness matrices and the necessary transformations derived, such that the element can be considered in an arbitrary shape and orientation with respect to is coordinate system. As another example, a curved beam element arbitrarily oriented in a reference coordinate system could be derived. In stepwise calculations using such generic elements the nodal and shape descriptions of the elements can be updated after each step, thereby dealing with a new set of elements for each new solution step. By suitably rotating the element local coordinate systems, the strain and stress increments can be referred to the same material fibers for each step and accumulated by summing indrements. This again is a case in which the nonlinear terms in the strain equations are retained only to determine the K_G matrix. Effectively, however, nonlinear strain behavior is obtained through the generic element updating, though of course, with accumulating error due to the stepwise linearization. It is noted that in general total strains and stresses are not obtainable with this method except through summing increments, due to the lack of a governing total strain equation. Thus, corrections for accumulating error cannot in general be made with this method, and this is a serious fault. It is also noted that the method is poorly adapted to large strain problems. Many nonlinear finite element programs, particularly those developed prior to the last five or ten years, have used this approach. It has been referred to as an Eulerian approach and as an updated Lagrangian approach, and refinements have been made to eliminate its inherent inaccuracies (Ref. 105).

If the finite element derivation retains the nonlinear strain equations not only for determining K_G, but also for determination of all strain, and hence stress, data, then it is said to be based on a total or incremental Lagrangian approach. There are many options for these purely Lagrangian approaches. Most approaches linearize the incremental strain, by dropping the second degree terms in the incremental strain equations, but retain access to the total strain as given by the nonlinear strain displacement equation. The result is similar in the individual steps to the generic element updating scheme described above, but with important advantages and disadvantages. An important advantage is that the availability of the total Lagrangian strain equations,

referred to the same coordinate systems as the incremental ones, makes it feasible to compute exact total strains, thereby avoiding the cumulative errors of summing linearized incremental values. This option is usually exercised by making periodic equilibrium corrections at the end of the computed increments. To accomplish this, total strains are used to compute element stresses and loads, and from these data residual, or error, loads are determined to be applied prior to, or during, the next increment. An important disadvantage of the total or incremental Lagrangian procedure is that it suffers from the role exchange of the displacement components which was described earlier and illustrated by Figure 1. It is noted that the generic element updating method avoids this difficulty due to having sequentially rotated the element coordinate system during the stepwise solution calculations.

Procedures such as the purely Lagrangian one, in which fully nonlinear strain equations are available, permit the development of finite element formulations in which the individual increments are not linearized. In this case the incremental force versus deflection equations are of quadratic (or higher) degree, and special solution procedures must be used to solve problems. Vos (Ref. 108) has developed such a procedure for a second degree nonlinear step behavior.

One further approach to nonlinear finite element formulation is discussed here. This approach combines the advantages of the generic element updating and the purely Lagrangian incrementation. It was developed to avoid the problems of displacement component role exchange and stepwise accumulating error, and it permits the use of simplified nonlinear strain equations adapted to beam, plate, and shell problems. This simplification, which need not necessarily be used, is the omission of all nonlinear strain terms except those associated with lateral (bending type) displacements. The procedure involves the updating of the element local coordinate system and transformation of element accumulated displacements, but not updating the generic form of the element. Thus, the accumulating deformations of the element are followed in a Lagrangian sense but are referred to a new element local system to start each step. Special coordinate transformation equations are required to transform the de-

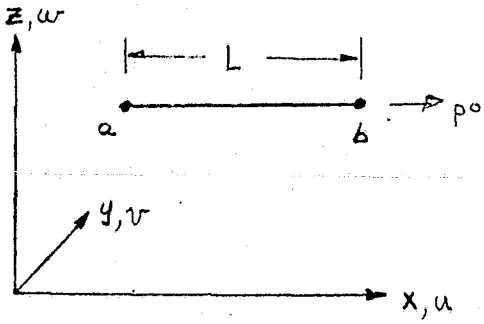
formations in this way, and through these transformations the calculation of exact, total Lagrangian strain is accomplished. This allows equilibrium checks and corrections to avoid errors due to accumulating linearized increments. The procedure is described in Volume II of this report.

All of the approaches briefly discussed above, and others as well, make use of the nonlinear strain equations in some way, but differ from one another in precisely how the nonlinear terms are incorporated into the calculations. Thus the form of the nonlinear strain-displacement equations used in the derivation of an element, and also the stiffness matrices of the element, are not definitive concerning the true nonlinear character of the element. To ascertain this, it is necessary to determine the details of the stepwise or iterative calculation procedure used with the element, and to evaluate carefully how the approximations in both the element derivation and the solution procedure affect its behavior.

6.0 ELEMENTS

ELEMENT ID: B-1

ELEMENT TYPE: Stringer or Truss Member



ASSUMED DISPLACEMENT SHAPE:

$$u = a_0 + a_1 x$$

$$v = b_0 + b_1 x$$

$$w = c_0 + c_1 x$$

DESCRIPTION:

This element is an axial load carrying member with 2 nodes and 3 DOF (u, v, w) per node. A stretching deformation state is accounted for. Only geometric nonlinearities are considered.

REFERENCE: [1], [2], [21], [64], [66], [69].

VARIATION OF THIS ELEMENT: [92], [93].

ADVANTAGES OR DISADVANTAGES:

ELEMENT ID: B-1

STRAIN DISPLACEMENT EQUATIONS:

$$\mathcal{E}_x = \mathcal{E}_{x^o} + \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2$$

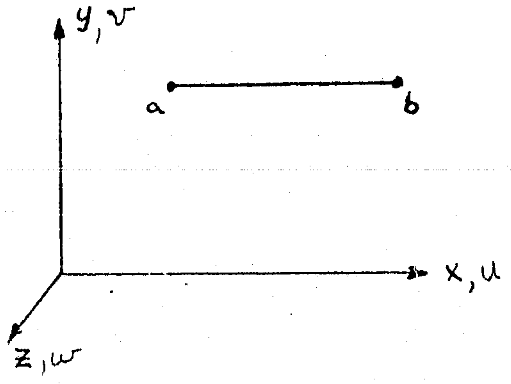
DISCUSSION:

This element is the simplest of the finite elements. It has been derived a number of times with few variations. A linear stress-strain relation is assumed and only quadratic terms in displacement gradients are retained in the strain energy. A Taylor series derivation is employed in Ref. [1]. The stiffness matrix obtained is termed the initial stress stiffness matrix and is reproduced below. A Lagrangian derivation is employed along with an incremental solution procedure.

$$[k^o]\{\delta\} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_a \\ u_b \end{Bmatrix}, \quad [k^1]\{\delta_2\} = \frac{P^o}{L}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} V_a \\ V_b \\ \omega_a \\ \omega_b \end{Bmatrix}$$

ELEMENT ID: B-2

ELEMENT TYPE: Straight Beam



ASSUMED DISPLACEMENT SHAPE:

$$u = a_0 + a x$$

$$v = b_0 + b_1 x + b_2 x^2 + b_3 x^3$$

DESCRIPTION:

This element is a straight beam of constant cross-section, with 2 nodes and 3 DOF (u, v, θ) per node. Stretching and bending in the x-z plane are accounted for (bending in the x-y plane is an elementary extension). Only geometric nonlinearities are considered.

REFERENCE: [2], [3], [4], [5], [15], [18], [21], [64], [66], [68], [88], [92], [96].

VARIATION OF THIS ELEMENT: [11], [14]

ADVANTAGES OR DISADVANTAGES:

Ref [6], ".... unsatisfactory for predicting post-buckling behavior although adequate for predicting buckling loads."

ELEMENT ID: B-2

STRAIN DISPLACEMENT EQUATIONS:

$$\varepsilon_x = \varepsilon_x^{\circ} + \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 - y\,\frac{\partial^2 v}{\partial x^2}$$

DISCUSSION:

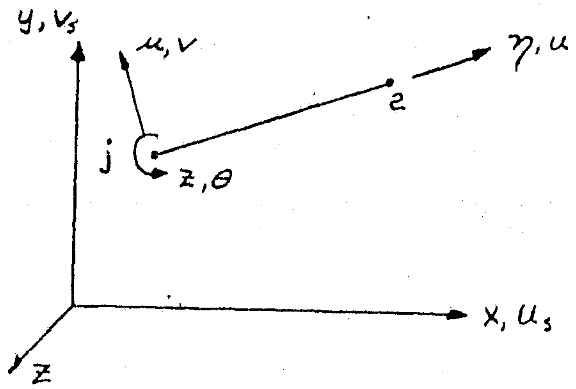
This element employs a linear stress-strain relation with only quadratic products of displacement gradients retained in the strain energy. A Lagrangian derivation is employed along with an incremental solution procedure. The initial stress stiffness matrix is reproduced below.

$$[k^1]\{\delta\} = \frac{P^{\circ}}{30L}\begin{bmatrix} 36 & & & \\ 3L & 4L^2 & & \\ -36 & -3L & 36 & \\ 3L & -L^2 & -3L & 4L^2 \end{bmatrix}\begin{Bmatrix} V_a \\ \theta_a \\ V_b \\ \theta_b \end{Bmatrix}$$

In Reference [3] this matrix is used in an eigenvalue buckling solution for beams of variable cross-section. In Reference [4] it is employed with the midpoint tangent incremental approach, and in Reference [5] it is used with nonlinearly elastic materials. In Reference [18] this matrix is called the stability coefficient matrix. In Reference [64] the nonlinear coupling between bending and torsion is included in the geometric stiffness matrix derivation. In addition, elastic, ideally plastic, linear strain hardening, or nonlinear strain hardening stress-strain laws may be used with the element. Viscoelastic effects are included in Ref. [68].

ELEMENT ID: B-2a

ELEMENT TYPE: Straight Beam



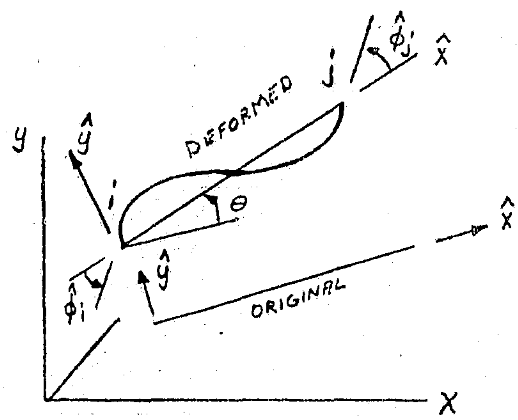
REFERENCE: [11], [37]

VARIATIONS FROM BASIC ELEMENT:

All combinations of displacement gradients are retainted in the strain energy yielding the K, N1 and N2 matrices. Various solution procedures are derived and examined in Reference [11]. Initial imperfections and an energy perturbation solution approach are used in Reference [37]. Only geometric nonlinearities are considered.

ELEMENT ID: B-2b

ELEMENT TYPE: Straight Beam



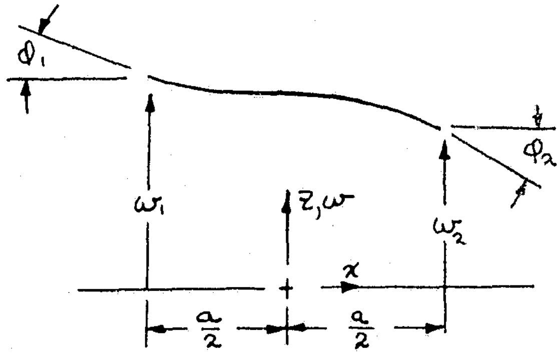
REFERENCE: [14]

VARIATIONS FROM BASIC ELEMENT:

This is a derivation using convected coordinates in the manner of Argyris. It is derived for large displacement, small strain problems with material nonlinearities. It uses a direct nodal force computational scheme.

Uses an unusual approach with convected coordinates which separates rigid body and deformational displacements through transformation. Lagrangian, Eulerian, and convected coordinates are all used. It is a total deformation rather than incremental deformation derivation, and it is fully nonlinear while still having linear strain-displacement equations. Uses direct, numerical integration node force evaluation. Does not make use of "initial stresses"; they do not appear in this derivation

ELEMENT ID: B-2c



ELEMENT TYPE: Straight Beam

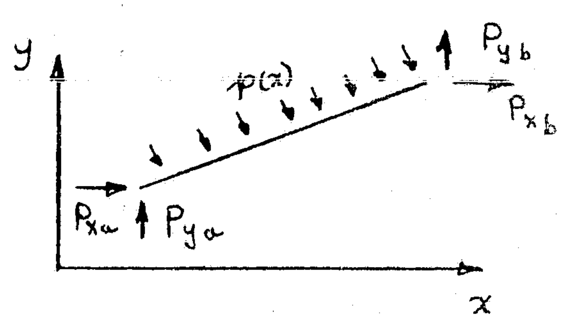
REFERENCE: [63]

VARIATIONS FROM BASIC ELEMENT:

This derivation employs the concept of natural loads and natural stiffness to formulate the large displacement problem. It is derived for large displacement, small strain problems using convected coordinates. An incremental solution procedure is developed. The method is applicable to post-buckling analyses where an iterative procedure is used to find the initial post-buckled position.

ELEMENT ID: B-2d

ELEMENT TYPE: Straight Beam



REFERENCE: [120]

VARIATIONS FROM BASIC ELEMENT:

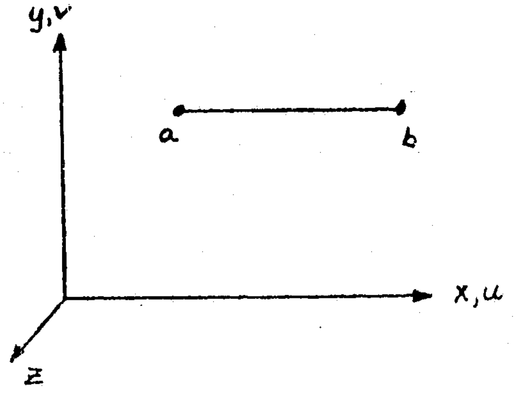
Author retains both $\frac{1}{2}\left(\frac{dv}{dx}\right)^2$ and $\frac{1}{2}\left(\frac{du}{dx}\right)^2$ strain terms, though the latter proves unimportant in his examples. Principal variation from the basic element is that the loads p(x) and P_x,P_y are permitted to change direction as functions of the deformation, producing "follower load" effects. With these effects represented by a matrix L, author's formulation has the form

$$\Delta Q = [[K] + [G] + [L]]\ \Delta q$$

where G is the geometric stiffness matrix and K is the stiffness matrix for linear behavior.

ELEMENT ID: B-3

ELEMENT TYPE: Straight Beam



ASSUMED DISPLACEMENT SHAPE:

$$u = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$$

$$v = b_0 + b_1 x + b_2 x^2 + b_3 x^3$$

DESCRIPTION:

This is the "stability" beam element where the quintic u displacement is consistent with the cubic v displacement from the equation of axial equilibrium. It has two nodes and 3 DOF (u, v, v_x) per node. Stretching and bending deformations are accounted for. Only geometric nonlinearties are considered. The excess generalized DOF are reduced on the elemental level.

REFERENCE: [6]

VARIATION OF THIS ELEMENT:

ADVANTAGES OR DISADVANTAGES:

Improved postbuckling behavior.

ELEMENT ID: B-3

STRAIN DISPLACEMENT EQUATIONS:

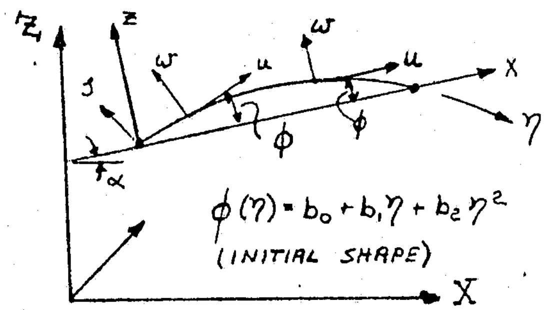
$$\varepsilon_x = \frac{\partial u}{\partial x} - y\frac{\partial^2 v}{\partial x^2} + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2$$

DISCUSSION:

This element employs a linear stress-strain relation with all displacement gradient combinations (up to quartic) retained in the strain energy. A Lagrangian derivation is employed.

ELEMENT ID: B-4

ELEMENT TYPE: Curved Beam



ASSUMED DISPLACEMENT SHAPE:

$$\begin{Bmatrix} u \\ w \\ \psi \end{Bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & (Z_1 - Z_{10})\cos(\phi+\alpha) - (X - X_0)\sin(\phi+\alpha) & \eta & 0 & 0 \\ -\sin\phi & \cos\phi & -(Z_1 - Z_{10})\sin(\phi+\alpha) - (X - X_0)\cos(\phi+\alpha) & 0 & \eta^2 & \eta^3 \\ 0 & 0 & 1 & \eta\frac{\partial\phi}{\partial\eta} & 2\eta & 3\eta^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix}$$

$$\psi = \frac{\partial w}{\partial \eta} + u\frac{\partial \phi}{\partial \eta}$$

DESCRIPTION:

This is a curved beam with 2 nodes and 3 DOF (u, w, ψ) per node. The rigid body modes are added explicitly to the strain-inducing modes. Stretching and bending deformation states are accounted for. Geometric and material nonlinearities are considered.

REFERENCE: [7], [12]

VARIATION OF THIS ELEMENT:

ADVANTAGES OR DISADVANTAGES:

ELEMENT ID: B-4

STRAIN DISPLACEMENT EQUATIONS:

$$\varepsilon = \varepsilon_o + \zeta K$$

$$\varepsilon_o = \frac{\partial u}{\partial \eta} - \omega \frac{\partial \phi}{\partial \eta} + \frac{1}{2}\left[\frac{\partial \omega}{\partial \eta} + u\left(\frac{\partial \phi}{\partial \eta}\right)\right]$$

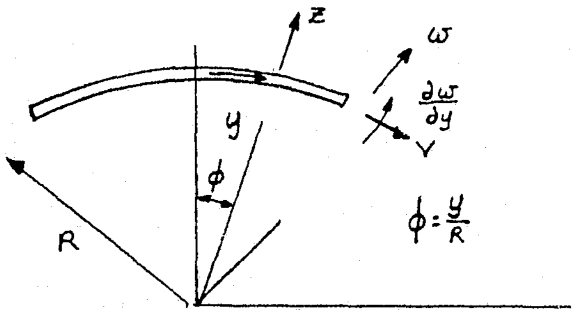
$$K = -\frac{\partial}{\partial \eta}\left[\frac{\partial \omega}{\partial \eta} + u\frac{\partial \phi}{\partial \eta}\right]$$

DISCUSSION:

The element is developed for an elastic-plastic, strain-hardening, and strain-rate material behavior through the use of the mechanical sublayer model. The element is developed for a transient response analysis and the form of the dynamic equations are not standard. Matrices are developed in terms of the products of stresses with linear and nonlinear strains to obtain well behaved numerical integration. A Lagrangian derivation is employed and all nonlinear products are retained in the Virtual Work expression.

ELEMENT ID: B-5

ELEMENT TYPE: Circular Arch



ASSUMED DISPLACEMENT SHAPE:

$$\begin{Bmatrix} v \\ w \end{Bmatrix} = \begin{bmatrix} \cos\left(\frac{y}{R}\right) & -\sin\left(\frac{y}{R}\right) & 1 & 0 & Ry & \frac{1}{2}Ry^2 \\ \sin\left(\frac{y}{R}\right) & \cos\left(\frac{y}{R}\right) & 0 & R & -R^2 & -R^2 y \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix}$$

DESCRIPTION:

This is a circular arch element with 2 nodes and 3 DOF (v, w, w_y) per node. A constant circumferential strain and a linearly varying bending strain is assumed. The deformation states arise from the simultaneous solution of these equations. Only geometric nonlinearities are considered.

REFERENCE: [8], [9]

VARIATION OF THIS ELEMENT:

ADVANTAGES OR DISADVANTAGES:

Improved convergence.

Probably not suitable for large element deflections or for post-buckling, due to constant circumferential strain

ELEMENT ID: B-5

STRAIN DISPLACEMENT EQUATIONS:

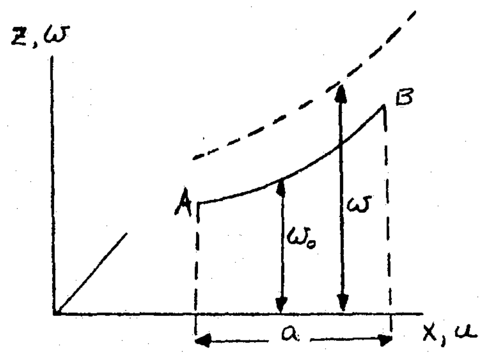
$$\varepsilon_y = \frac{\partial v}{\partial y} + \frac{w}{R} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - z\left(\frac{\partial^2 w}{\partial y^2} - \frac{1}{R}\frac{\partial v}{\partial y}\right)$$

DISCUSSION:

In Reference [8] the linear element of Reference [9] is extended to geometrically nonlinear problems. Rigid body states are explicitly formed because of the integration of the assumed strain states. The rational for the choice of strain states is apparently empirical. All combinations of displacement derivatives are retained in the strain energy. The Lagrangian strains are employed. The element is used in the linearized incremental method based on mid-increment stiffness and Newton-Rapheon iteration.

ELEMENT ID: B-6

ELEMENT TYPE: Shallow Curved Beam



ASSUMED DISPLACEMENT SHAPE:

$$u = [0, 0, 0, 0, 1, x, x^2, x^3] \{\alpha\}$$
$$w = [1, x, x^2, x^3, 0, 0, 0, 0] \{\alpha\}$$

w_o = Not specified, needs to be dealt with in matrix generation, by integrations.

DESCRIPTION:

This is a shallow curved beam with 2 nodes and 4 DOF per node (u, du/dx, w, dw/dx). An initial imperfection, w_o, is used to specify the initial curved geometry. Stretching and bending deformation states are accounted for. Only geometric nonlinearities are considered.

REFERENCE: [10]

VARIATION OF THIS ELEMENT: [12]

ADVANTAGES OR DISADVANTAGES:

ELEMENT ID: B-6

STRAIN DISPLACEMENT EQUATIONS:

$$\mathcal{E}_x = \frac{\partial u}{\partial x} + Z\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w_o}{\partial x^2}\right) + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - \frac{1}{2}\left(\frac{\partial w_o}{\partial x}\right)^2$$

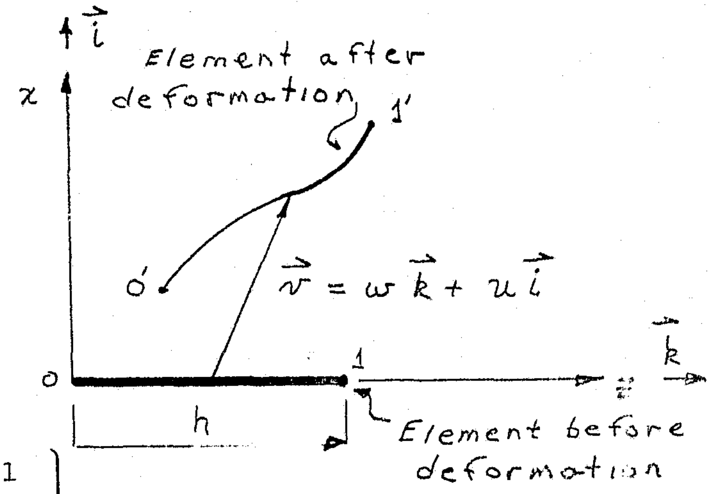
(not explicitly presented)
(potential energy method used)

DISCUSSION:

This element employs a linear stress-strain relation with only the quadratic combinations of displacement gradients retained in the strain energy, thus yielding the classical k' matrix. It should be noted that retaining the displacement gradient du/dx as a nodal DOF does not conform with the usual element assemblage procedure. A Lagrangian derivation is employed. The mid-increment solution procedure is proposed.

ELEMENT ID: B-6a

ELEMENT TYPE: BEAM (For in-plane large displacement analysis)



REFERENCE: [72]

VARIATIONS FROM BASIC ELEMENT:

$$\begin{Bmatrix} u \\ w \end{Bmatrix} = \begin{bmatrix} u_o & u'_o & u_1 & u'_1 \\ w_o & w'_o & w_1 & w'_1 \end{bmatrix} \begin{Bmatrix} 2\zeta^3 - 3\zeta^2 + 1 \\ (\zeta^3 - 2\zeta^2 + \zeta)h \\ -2\zeta^3 + 3\zeta^2 \\ (\zeta^3 - \zeta^2)h \end{Bmatrix}$$

where $\zeta = z/h$

$(\;)' = \frac{d}{dz}(\;)$

o : left end

1 : right end

DESCRIPTION: 2 nodes, 4 degrees of freedom per node – (Stretching and bending deformations are modelled). Displacement components are referred to the initial configuration.

ELEMENT ID: B-6a

ADVANTAGES OR DISADVANTAGES: The cubic polynomial representation for both displacement components, allows for the use of this element with arbitrarily large displacements and rotations. (The use of a linear representation for the z-component would lead, on the contrary, to a non-objective curvature measure ruling out the use of such an element for cases of large rotations).

STRAIN DISPLACEMENT EQUATIONS:

$$e = w' + \frac{1}{2}(w'^2 + u'^2)$$

$$\chi = u''(1 + w') - w'' u'$$

where e: axial stretching measure

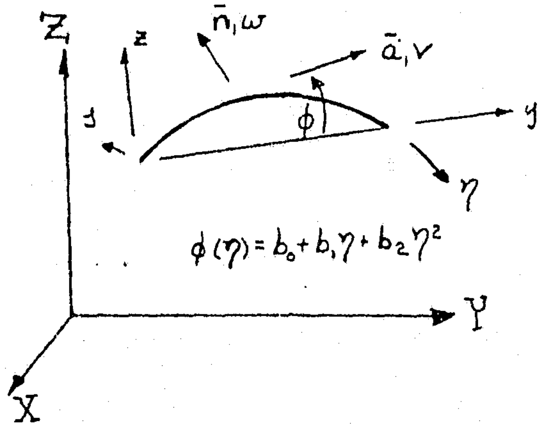
χ: axial bending (curvature) measure

DISCUSSION: The element is used in connection with an exact non-linear formulation analogous to Budiansky's non-linear Shell Theory (ASME T. of Appl. Mech., June 1968). The strain-energy expression used is equivalent to the 3-D expression under Kirchhoff's hypothesis. Constitutive equations may be prescribed in terms of Kirchhoff's stress tensor and Green's strain tensor and integrated over the cross-section.

Resulting equations are polynomials of the 3rd degree in the nodal displacements and their first z-derivatives. Preferred solution procedure: Newton-Raphson.

ELEMENT ID: B-7

ELEMENT TYPE: Curved Beam Element



ASSUMED DISPLACEMENT SHAPE:

$$\begin{Bmatrix} v \\ w \end{Bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & -z\cos\phi + y\sin\phi & \eta & 0 & 0 & \eta^2 & \eta^3 \\ -\sin\phi & \cos\phi & z\sin\phi + y\cos\phi & 0 & \eta^2 & \eta^3 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \beta_1 \\ \vdots \\ \beta_8 \end{Bmatrix}$$

DESCRIPTION:

This is a curved beam with 2 nodes and 4 DOF ($v, w, \psi = \partial w/\partial\eta - v/R$, and $\delta = \partial v/\partial\eta - w/R$) per node. The rigid body modes are added explicitly to the strain inducing modes. Stretching and bending deformation states are accounted for, i.e., a Timoshenko type element. Geometric and material nonlinearities are considered.

REFERENCE: [12]

VARIATION OF THIS ELEMENT: [10]

ADVANTAGES OR DISADVANTAGES:

Improved strain prediction.

ELEMENT ID: B-7

STRAIN DISPLACEMENT EQUATIONS:

$$\varepsilon = \varepsilon_o + \zeta k$$

$$\varepsilon_o = \left(\frac{\partial v}{\partial \eta} + \frac{w}{R}\right) + \frac{1}{2}\left(\frac{\partial w}{\partial \eta} - \frac{V}{R}\right)^2$$

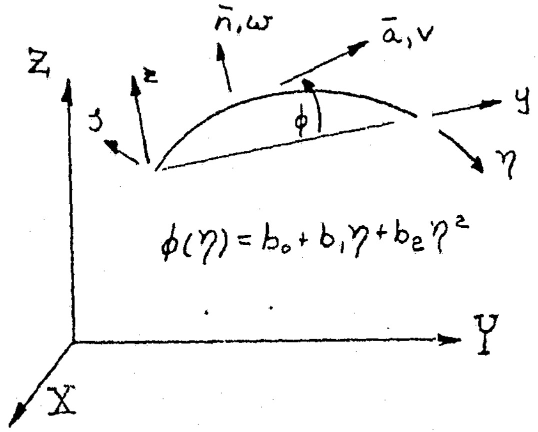
$$K = -\frac{\partial}{\partial \eta}\left(\frac{\partial w}{\partial \eta} - \frac{V}{R}\right)$$

DISCUSSION:

The element is developed for an elastic plastic, strain hardening, and strain rate dependent material behavior through the use of the mechanical sublayer model. The element is developed for a transient response analysis. All nonlinear products are retained in the virtual work expression. The author states the fourth DOF, δ, at each node is retained because of a lack of a rational condensation process for dynamic problems.

ELEMENT ID: B-8

ELEMENT TYPE: Curved Beam Element



ASSUMED DISPLACEMENT SHAPE:

$$\begin{Bmatrix} v \\ \omega \\ \gamma_0 \end{Bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & -z\cos\phi + y\sin\phi & \eta & \eta^2 & 0 & 0 & 0 \\ -\sin\phi & \cos\phi & z\sin\phi + y\cos\phi & 0 & 0 & \eta^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \eta \end{bmatrix} \begin{Bmatrix} \beta_1 \\ \vdots \\ \beta_8 \end{Bmatrix}$$

DESCRIPTION:

This is a curved beam with 3 nodes and 3 DOF (v, w, θ) at the two end nodes and 2 DOF (v, w) at the midpoint node. The rigid body modes are added explicitly to the strain inducing modes. Stretching, bending, and shearing deformation states are accounted for. Geometric and material nonlinearities are considered.

REFERENCE: [12]

VARIATION OF THIS ELEMENT:

ADVANTAGES OR DISADVANTAGES:

ELEMENT ID: B-8

STRAIN DISPLACEMENT EQUATIONS:

$$\varepsilon(\eta,\zeta) = \varepsilon_o(\eta) + \zeta k(\eta), \qquad \gamma(\eta,\zeta) = \gamma_o(\eta)$$

$$\varepsilon_o = \left(\frac{\partial v}{\partial \eta} + \frac{w}{R}\right) + \frac{1}{2}\left(\frac{\partial w}{\partial \eta} - \frac{v}{R}\right)^2$$

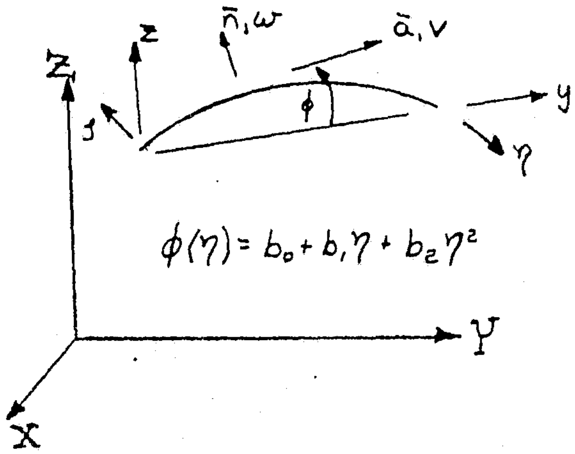
$$K = \frac{\partial \theta}{\partial \eta}, \quad \gamma_o = \left(\frac{\partial w}{\partial \eta} - \frac{v}{R}\right) + \theta$$

DISCUSSION:

The element is developed for an elastic-plastic, strain hardening, and strain rate dependent material behavior through the use of the mechanical sublayer model. The element is developed for a transient response analysis. All nonlinear products are retained in the virtual work expression.

ELEMENT ID: B-9

ELEMENT TYPE: Curved Beam Element



ASSUMED DISPLACEMENT SHAPE:

$$\begin{Bmatrix} V \\ \omega \\ \theta \end{Bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & -Z\cos\phi + y\sin\phi & \eta & 0 & 0 \\ -\sin\phi & \cos\phi & Z\sin\phi + y\cos\phi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \eta \end{bmatrix} \begin{Bmatrix} \beta_1 \\ \vdots \\ \beta_6 \end{Bmatrix}$$

DESCRIPTION:

This is a curved beam with 2 nodes and 3 DOF (v, w, Θ) per node. The rigid body modes are added explicitly to the strain inducing modes. Stretching, bending, and shearing deformation states are accounted for. Geometric and material nonlinearities are considered.

REFERENCE: [12]

VARIATION OF THIS ELEMENT: [16]

ADVANTAGES OR DISADVANTAGES:

Requires small element size to accurately obtain load output.

ELEMENT ID: B-9

STRAIN DISPLACEMENT EQUATIONS:

$$\varepsilon(\eta,\zeta) = \varepsilon_o(\eta) + \zeta k(\eta), \qquad \gamma(\eta,\zeta) = \gamma_o(\eta)$$

$$\varepsilon_o = \left(\frac{\partial v}{\partial \eta} + \frac{w}{R}\right) + \frac{1}{2}\left(\frac{\partial w}{\partial \eta} - \frac{v}{R}\right)^2$$

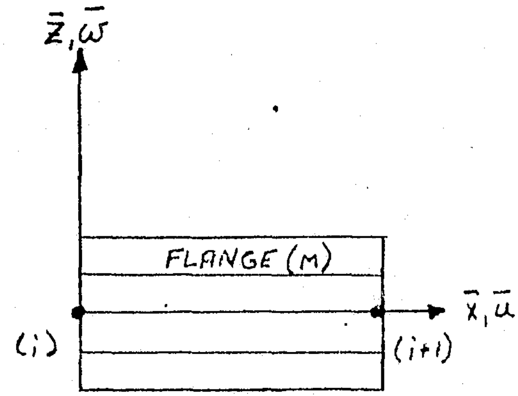
$$K = \frac{\partial \theta}{\partial \eta}, \qquad \gamma_o = \left(\frac{\partial w}{\partial \eta} - \frac{v}{R}\right) + \theta$$

DISCUSSION:

The element is developed for an elastic-plastic, strain-hardening, and strain rate dependent material behavior through the use of the mechanical sublayer model. The element is developed for a transient response analysis. All nonlinear products are retained in the virtual work expression.

ELEMENT ID: B-9a

ELEMENT TYPE: Straight Beam



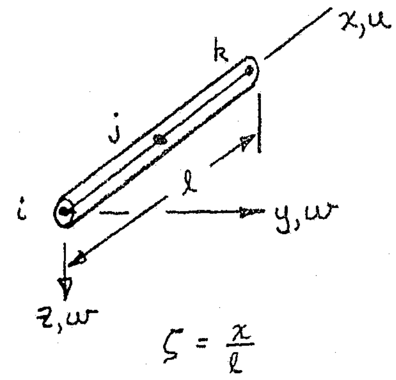
REFERENCE: [16]

VARIATIONS FROM BASIC ELEMENT: B-9

This is also a mechanical sublayer model, but the element is generated from an assembly of flanges and shear webs and not from energy considerations. The element is developed for an elastic - perfectly plastic material and a transient response analysis. Geometric nonlinearities are accounted for by updating coordinates.

ELEMENT ID: B-10

ELEMENT TYPE: Stringer or Truss Member



ASSUMED DISPLACEMENT SHAPE:

$$\begin{Bmatrix} u(x) \\ v(x) \\ w(x) \end{Bmatrix} = (2\zeta^2 - 3\zeta + 1) \begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix} + (2\zeta^2 - \zeta) \begin{Bmatrix} u_j \\ v_j \\ w_j \end{Bmatrix} + 4(\zeta - \zeta^2) \begin{Bmatrix} u_k \\ v_k \\ w_k \end{Bmatrix}$$

Initial strain distribution ε = constant

DESCRIPTION:

This element is a linear strain, axial load carrying member with 3 nodes and 3 DOF (u,v,w) per node. A stretching deformation state is accounted for. Both geometric and material nonlinearities are considered.

REFERENCE: [64]

VARIATION OF THIS ELEMENT:

ADVANTAGES OR DISADVANTAGES:

ELEMENT ID: B-10

STRAIN DISPLACEMENT EQUATIONS:

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2$$

DISCUSSION:

This element employs a linear elastic stress-strain relation with only quadratic products of displacement gradients retained in the strain energy. A Lagrangian derivation is employed to yield the geometric stiffness matrix $[k^1]$. The stiffness and geometric stiffness matrices are reproduced below (where $\overline{F}$ is the average axial force). Apparently the updated or convected coordinate approach to the solution of the geometrically nonlinear problem is used. Plastic strains are calculated using a uniaxial bilinear stress-strain relation or a nonlinear Ramberg-Osgood relation. Perfect plasticity is also accommodated.

$$[k^\circ]\{\delta\} = \frac{AE}{3\ell}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}\begin{Bmatrix} u_i \\ u_j \\ u_k \end{Bmatrix}$$

$$[k']\{\overline{\delta}\} = \frac{\overline{F}}{\ell}\begin{bmatrix} 7 & 1 & -8 & 0 & 0 & 0 \\ 1 & 7 & -8 & 0 & 0 & 0 \\ -8 & -8 & 16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 & 1 & -8 \\ 0 & 0 & 0 & 1 & 7 & -8 \\ 0 & 0 & 0 & -8 & -8 & 16 \end{bmatrix}\begin{Bmatrix} v_i \\ v_j \\ v_k \\ w_i \\ w_j \\ w_k \end{Bmatrix}$$

HANDBOOK OF NONLINEAR

FINITE ELEMENTS

ELEMENT ID: B-11

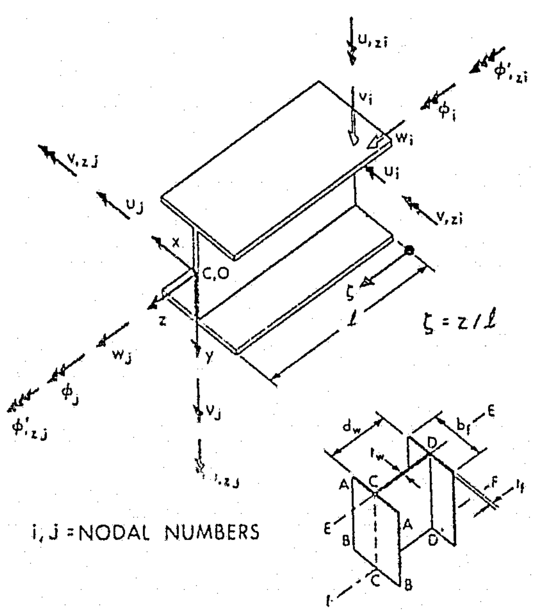
ELEMENT TYPE: Beam (Linear elastic)

ASSUMED DISPLACEMENT SHAPE:

1. Cubic polynomial for u displacements.
2. Cubic polynomial for v displacements.
3. Cubic polynomial for torsional displacements.
4. Linear polynomial for axial displacements.
5. Cubic polynomial for flange displacements.

DESCRIPTION:

(a) For member buckling the element has 14 degrees of freedom associated with the nodal variables shown in Fig. 1.



$$\langle \underline{u} \rangle = \langle u_i,\ \ell u_{,zi},\ u_j,\ \ell u_{,zj} \rangle$$

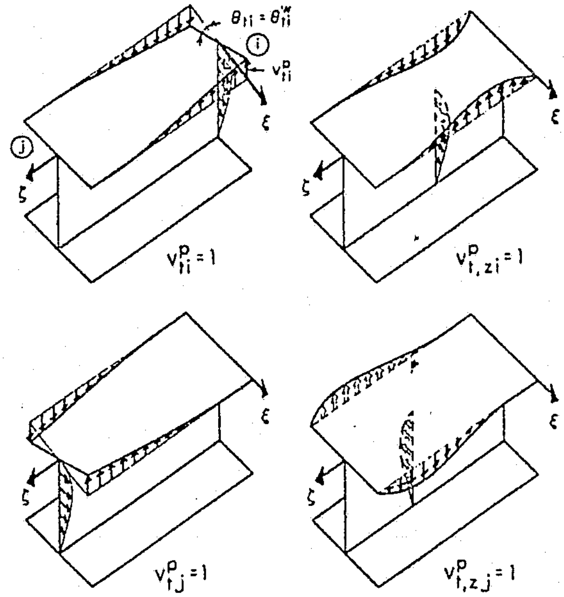
$$\langle \underline{v} \rangle = \langle v_i,\ \ell v_{,zi},\ v_j,\ \ell v_{,zj} \rangle$$

$$\langle \underline{\phi} \rangle = \langle \phi_i,\ \ell \phi_{,zi},\ \phi_j,\ \ell \phi_{,zj} \rangle$$

$$\langle \underline{w} \rangle = \langle w_i,\ w_j \rangle$$

Figure 1 - Beam Element and Nodal Displacements

(b) For local buckling the element has 8 degrees of freedom, four associated with the top flange (as illustrated in Fig. 2), and four associated with the bottom flange.



$$< \underline{v}_t^p > \; = \; < v_{ti}^p , \; \ell v_{t,zi}^p , \; v_{tj}^p , \; \ell v_{t,zj}^p >$$

$$< \underline{v}_b^p > \; = \; < v_{bi}^p , \; \ell v_{b,zi}^p , \; v_{bj}^p , \; \ell v_{b,zj}^p >$$

$$< \underline{\ell} > \; = \; < \; < \underline{v}_t^p > , \; < \underline{v}_b^p > \; >$$

Figure 2 - Plate Flexure Shape Functions and Nodal Displacements

The element is formulated with respect to the initial coordinate system (Lagrangian formulation).

REFERENCE: [70]

VARIATIONS OF THIS ELEMENT:

Considering only the (a) displacement patterns the element can be used for a linear eigenvalue member stability analysis which couples axial, torsional and biaxial bending effects. Introducing the (b) displacement pattern allows local flange buckling to be coupled to the behavior in (a). When (b) displacement patterns only are used, an uncoupled flange buckling analysis may be performed.

ADVANTAGES OR DISADVANTAGES:

Beam buckling problems or load deformation problems involving three dimensional response may be modelled with a minimum number of degrees of freedom.

STRAIN DISPLACEMENT EQUATIONS:

(a) For member displacements,

$$u = u_o - (y - e_y)\ \phi$$

$$v = v_o + (x - e_x)\ \phi$$

$$w = w_c - y\ v_o' - x\ u_o' + \bar{\omega}_n\ \phi$$

$$\varepsilon_z = \frac{\partial w}{\partial z}$$

in which, u_o and v_o are displacements of the shear center; w_c is the displacement of the centroid (origin of x - y coordinates); and ϕ is the twist.

(b) For local plate displacements,

$$u^* = - z w^*_{;\alpha}$$

$$v^* = - z w^*_{;\beta}$$

in which w^* is the traverse local plate displacement, α and β are local coordinate axes, and z is the normal coordinate measured from the mid-surface of the plate.

DISCUSSION:

Stress-strain response is assumed linear elastic. Derivation is by the principle of virtual displacements with the majority of terms in the "geometric stiffness" matrix being retained. Solution may be either a linear eigenvalue solution or a nonlinear load-deformation solution. Lagrangian coordinates are used throughout.

HANDBOOK OF NONLINEAR FINITE ELEMENTS

ELEMENT ID: B-12

ELEMENT TYPE: Inelastic beam (Shanley tangent modulus)

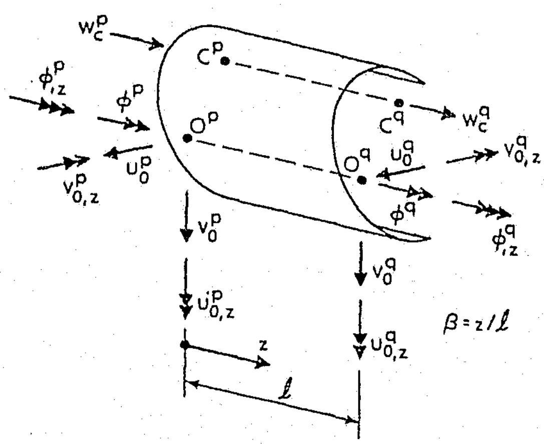
ASSUMED DISPLACEMENT SHAPE:

1. Cubic polynomial for u displacement.
2. Cubic polynomial for v displacement.
3. Cubic polynomial for torsional displacement.
4. Quadratric polynomial for axial displacement.

DESCRIPTION:

The element has two nodes at each end, one for u, v and ϕ nodal degrees of freedom, and one for a w degree of freedom (see Figure 1). In addition there is a midpoint node for an additional w degree of freedom which permits a quadratic polynomial to be used for axial displacements. This degree of polynomial is the minimum permissable to obtain a balance of nodal forces at interface nodes when inelastic deformations occur.

The element tangent stiffness has been formulated by employing transformed section properties. A trilinear stress strain curve, as shown on Fig. 2, allows subdivision of each component plate segment into at most five regions, as shown in Fig. 3, which may be assembled into a section of equivalent stiffness as shown in Fig. 4.



$$\{\underline{u}\}^T = < u_0^p,\ \ell u_{0,z}^p,\ u_0^q,\ \ell u_{0,z}^q >$$

$$\{\underline{v}\} = < v_0^p,\ \ell v_{0,z}^p,\ v_0^q,\ \ell v_{0,z}^q >$$

$$\{\underline{\phi}\} = < \phi^p,\ \ell \phi_z^p,\ \phi^q,\ \ell \phi_z^q >$$

$$\{\underline{w}\} = < w_c^p,\ w_c^{(p+q)/2},\ w_c^q >$$

$$\{r^e\}^T = < \{\underline{u}\}^T, \{\underline{v}\}^T, \{\underline{\phi}\}^T, \{\underline{w}\}^T >$$

Figure 1 - Finite Element Degrees-of-Freedom and Nodal Vectors

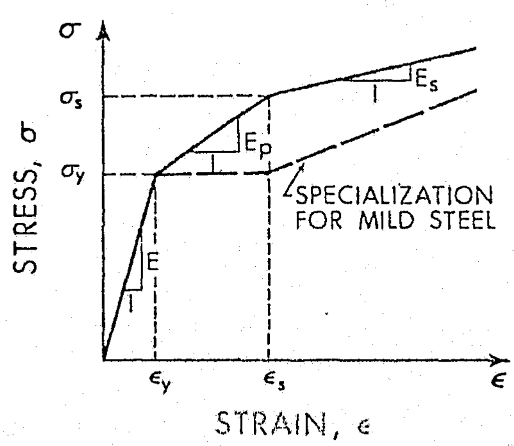


Figure 2 - Trilinear Stress-Strain Diagram

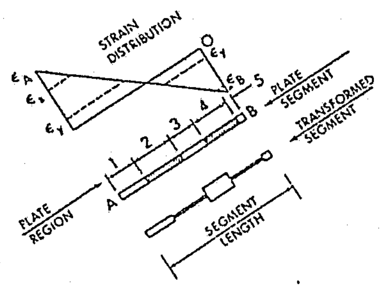


Figure 3 - Transformed Section of Plate Segment

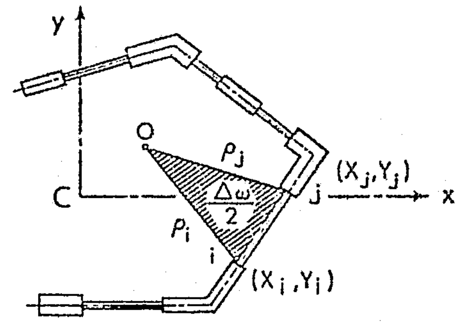


Figure 4 - Arbitrary Transformed Section

REFERENCE: [71]

ADVANTAGES OR DISADVANTAGES:

The formulation refers element properties to a fixed set of axes so that incremental element matrices do not require a transformation prior to assembly.

STRAIN DISPLACEMENT EQUATIONS:

$$u = u_o - (y - e_y)\,\phi$$

$$v = v_o + (x - e_x)\,\phi$$

$$w = w_c - y\,v_o' - x\,u_o' + \overline{\omega_n}\,\phi$$

$$\varepsilon_z = \frac{\partial w}{\partial z}$$

or

$$\varepsilon_z = \frac{\partial w}{\partial z} + \frac{1}{2}\left\{ \left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 \right\}$$

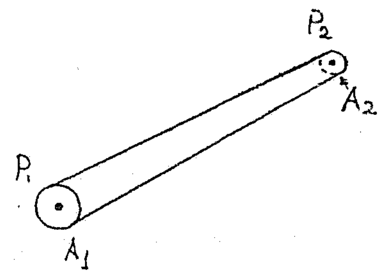
in which u_o and v_o are displacements of an arbitrary 'shear center' axis; w_c is the displacement of an arbitrary 'centroid axis' (origin of x - y coordinates); and ϕ is the twist of the cross-section.

DISCUSSION:

Derivation of equilibrium equations is by the principle of virtual work and retains terms significant in the geometric stiffness matrix. An iterative Newton Raphson procedure is used for solution but equilibrium is checked against the total applied load, thus preventing drift from the true solution. The program can be used for an eigenvalue solution to give inelastic buckling loads or for a nonlinear load deflection analysis to give inelastic instability loads. Lagrangian coordinates are used throughout.

ELEMENT ID: B-13

ELEMENT TYPE: FLA 2 (normal stress flange in 3-space)



ASSUMED DISPLACEMENT SHAPE: linear

DESCRIPTION:

Number of nodes : 2
Degrees of freedom : u, v, w at each node, all together 6
Deformation type : stretching
Isotropic

REFERENCE: [92], [93]

VARIATION OF THIS ELEMENT: [1], [2], [21], [64], [66], [69].

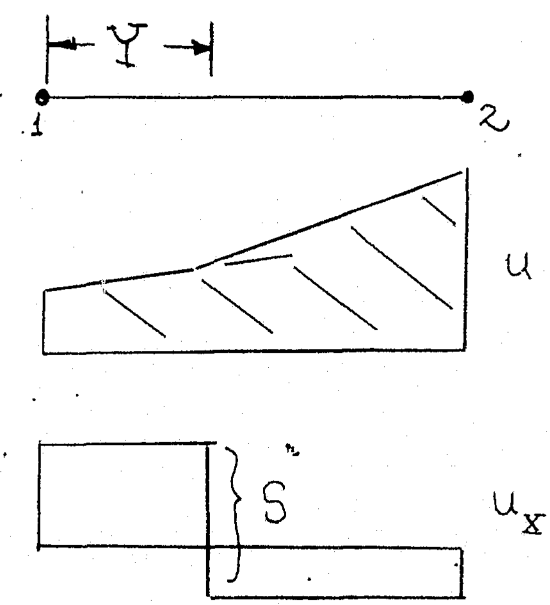
ADVANTAGES OR DISADVANTAGES:

DISCUSSION: Based on the concept of natural stress invariants for large rotations.

ELEMENT ID: B-14

ELEMENT TYPE: Elastic rod

ASSUMED DISPLACEMENT SHAPE: Piece-wise linear; contains built in discontinuities for shock wave calculations.



DESCRIPTION:
No. of nodes: 2
Degrees of freedom: 4
Coordinate system: Cartesian
Isotropic materials

REFERENCE:
[110], [111], [112], [113], [114].

VARIATION OF THIS ELEMENT:

ADVANTAGES OR DISADVANTAGES:

Allows for sharp definition of shock fronts without an undesirable degree of artificial dissipation.

Strain Displacement Equations:

$$\gamma_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{m,i}u_{m,j})$$

Discussion:

1. Stress-Strain equations: $\sigma^{ij} = \frac{1}{2}(\frac{\partial W}{\partial \gamma_{ij}} + \frac{\partial W}{\partial \gamma_{ji}}) + pG^{ij}$

2. Terms retained in the strain energy: All

3. Derivations from other than strain energy considerations: Galerkin Methods, Conservation of Energy.

4. Unique derivation techniques: See (c) above.

5. Lagrangian, Eulerian, convected: Lagrangian (convected on material)

6. Numerical integration schemes:

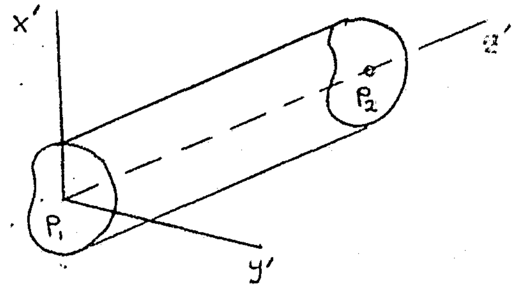
   For static problems: Incremental Loading (Euler's Method) plus Newton-Raphson corrections.

7. Required or preferred solution procedure:

   For dynamic problems: Incremental Loading. Divided central difference methods plus a Lax-Wendroff method.

ELEMENT ID: B-15

ELEMENT TYPE: BECOS (BECOSX)
(solid beam of constant cross section in 3 space)



ASSUMED DISPLACEMENT SHAPE: third order

Figure:

DESCRIPTION:

Number of nodes	:	2, located on centroid of end cross-sections
Degrees of freedom	:	u, v, w, ϕ_x, ϕ_y, ϕ_z at each node, all together 12
Deformation type	:	stretching, bending without shear effects, St. Venant Torsion
Isotropic		

REFERENCE: [92], [96].

VARIATION OF THIS ELEMENT: [2], [3], [4], [5], [15], [18], [21], [64], [66], [68], [88]. The concept was also applied to beams connected to eccentric nodal points (BECOSX).

ADVANTAGES OR DISADVANTAGES:

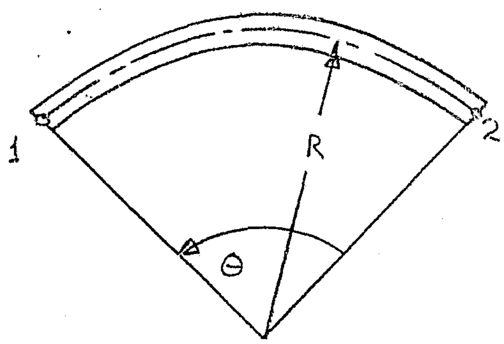
Strain-Displacement Equation and Discussion: The original derivation of this element is based on the natural stress invariants for large rotations. One special version started also with nonlinear strain-displacement relations (large rotations).

ELEMENT ID: B-16

ELEMENT TYPE: CIRCA
(circular beam in space)

ASSUMED DISPLACEMENT SHAPE: Third order

Figure:



DESCRIPTION:

Number of nodes : 2

Degrees of freedom: u, v, w, ϕ_x, ϕ_y, ϕ_z at any nodal point, all together 12

Deformation type : stretching, bending torsion (St. Venant).

REFERENCE: [96]

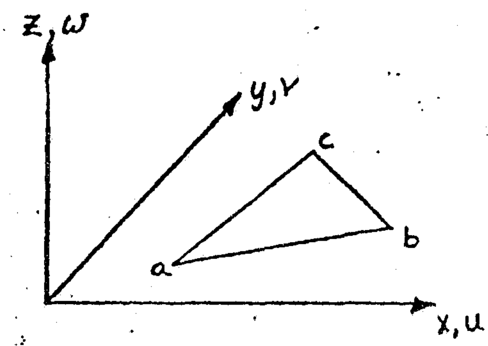
VARIATION OF THIS ELEMENT:

ADVANTAGES OR DISADVANTAGES:

DISCUSSION: Based on the concept of natural stress invariants for large rotations.

ELEMENT ID: P-1

ELEMENT TYPE: Membrane Triangular Plate



ASSUMED DISPLACEMENT SHAPE:

$$u = a_0 + a_1 x + a_2 y$$

$$v = b_0 + b_1 x + b_2 y$$

$$\omega = c_0 + c_1 x + c_2 y$$

DESCRIPTION:

This element is a plane stress triangular plate with 3 nodes and 3 DOF (u, v, w) per node. Only the stretching deformation state is accounted for. Only geometric nonlinearities are considered.

REFERENCE: [1], [2], [21], [60]
[107], [113], [114], [115], [117], [118], [94].

VARIATION OF THIS ELEMENT:

ADVANTAGES OR DISADVANTAGES:

Fully compatible element

ELEMENT ID: P-1

STRAIN DISPLACEMENT EQUATIONS:

$$\varepsilon_x = \varepsilon_x^o + \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2$$

$$\varepsilon_y = \varepsilon_y^o + \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2$$

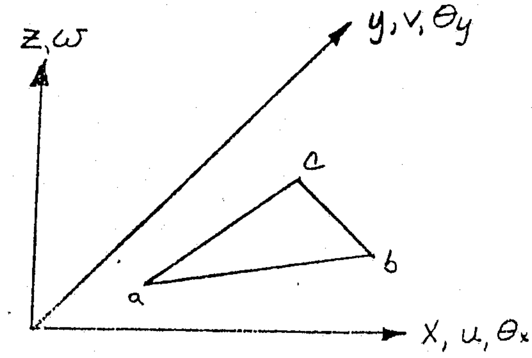
$$\gamma_{xy} = \gamma_{xy}^o + \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}$$

DISCUSSION:

This is another of the simplex elements with an assumed linear displacement state which results in a constant stress, constant strain element. A linear stress-strain relation is assumed and only quadratic terms are retained in the strain energy. A Lagrangian derivation is employed along with an incremental solution procedure. Node force increments are determined from stiffness matrices.

ELEMENT ID: P-1a

ELEMENT TYPE: Membrane and Bending Triangular Plate



REFERENCE: [20], [21]

VARIATIONS FROM BASIC ELEMENT: P-1

This element includes bending and shearing deformations in addition to the stretching deformation states described in P-1. The additional assumed displacement states are:

$$\theta_x = d_0 + d_1 x + d_2 y$$

$$\theta_y = e_0 + e_1 x + e_2 y$$

where

$$\bar{u} = u(x,y) + z\theta_y(x,y)$$

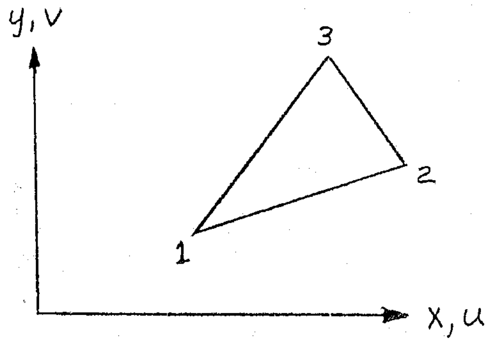
$$\bar{v} = v(x,y) - z\theta_x(x,y)$$

Only geometric nonlinearities are accounted for. The derivation is generalized to include orthotropic materials.

Degrees of freedom per node are u, v, w, θ_x, θ_y.

ELEMENT ID: P-1b

ELEMENT TYPE: Membrane Triangular Plate



REFERENCE: [19]

VARIATIONS FROM BASIC ELEMENT: P-1

This derivation considers material nonlinearities in conjunction with the basic isotropic membrane plate. Strains are assumed to be of the form

$$\varepsilon_x = \varepsilon_{x_e} + \alpha T + \varepsilon_{x_{pt}}$$

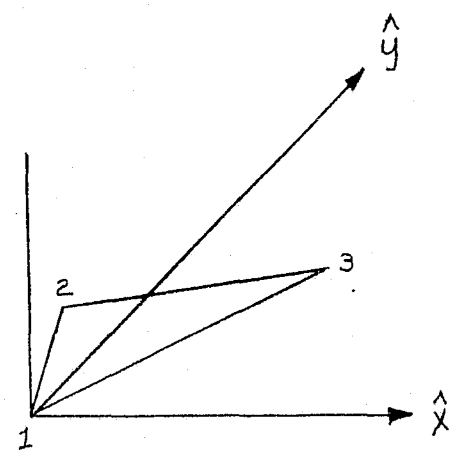
$$\varepsilon_y = \varepsilon_{y_e} + \alpha T + \varepsilon_{y_{pt}}$$

$$\gamma_{xy} = \gamma_{xy_e} + \gamma_{xy_{pt}}$$

where the subscripts e and pt denote elastic and inelastic respectively. The inelastic strains include both creep and time-independent plastic strains. The inelastic strains are converted to a column matrix of loads. An incremental solution procedure is employed.

ELEMENT ID: P-1c

ELEMENT TYPE: Membrane Triangular Plate



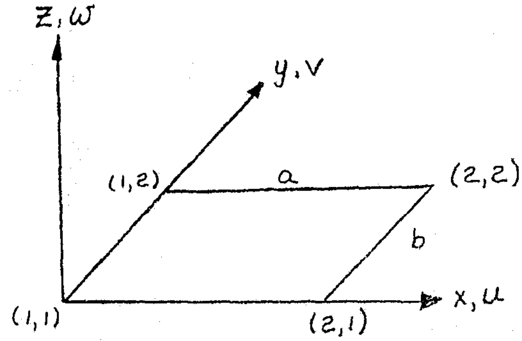
REFERENCE: [14]

VARIATIONS FROM BASIC ELEMENT: P-1

This derivation considers large rotations in the plane of the element. Convected coordinates are used which separate rigid body and deformation displacements through transformations. Lagrangian, Eulerian, and convected coordinates are all used. It is a total deformation rather than an incremental deformation derivation, and it is fully nonlinear, yet having linear strain-displacement equations. It uses direct numerical integration node force evaluation. It is derived for large displacement, small strain problems with material nonlinearities.

ELEMENT ID: P-2

ELEMENT TYPE: Rectangular Plate



ASSUMED DISPLACEMENT SHAPE:

$$w(x,y) = \sum_{i=1}^{2} \sum_{j=1}^{2} \left[H_{0i}^{(1)}(x)\, H_{0j}^{(1)}(y)\, w_{ij} + H_{1i}^{(1)}(x)\, H_{0j}^{(1)}(y)\, w_{x_{ij}} + H_{0i}^{(1)}(x)\, H_{1j}^{(1)}(y)\, w_{y_{ij}} + H_{1i}^{(1)}(x)\, H_{1j}^{(1)}(y)\, w_{xy_{ij}} \right]$$

$$H_{01}^{(1)}(x) = (2x^3 - 3ax^2 + a^3)/a^3 \qquad H_{11}^{(1)}(x) = (x^3 - 2ax^2 + a^2x)/a^2$$

$$H_{02}^{(1)}(x) = -(2x^3 - 3ax^2)/a^3 \qquad H_{12}^{(1)}(x) = (x^3 - ax^2)/a^2$$

Similarly for y direction replace x by y, and a by b. Similar expressions for u and v.

DESCRIPTION:

This element is an isotropic rectangular plate with 4 nodes and 12 DOF, (u, v, w, u_x, v_x, w_x, u_y, v_y, w_y, u_{xy}, v_{xy}, w_{xy}) per node. Stretching and bending deformation states are accounted for. Displacements are approximated by products of one dimensional cubic Hermite interpolation polynomials. Only geometric nonlinearities are considered.

REFERENCE: [16]

VARIATION OF THIS ELEMENT: See also P-12, P-2a, S-5

ADVANTAGES OR DISADVANTAGES:

The bicubic interpolation functions for u, v, and w admit to six linearly independent displacement states of very little strain energy. Consequently, this element offers a better resolution of the rigid body mode problem. The addition of the w_{xy} terms to the transverse deflection corrects the deficiency of the Papperfuss element [28] in allowing a constant twist state. The selected nodal DOF may not be compatible with other elements in a general purpose structural analysis computer program.

ELEMENT ID: P-2

STRAIN DISPLACEMENT EQUATIONS:

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - z\left(\frac{\partial^2 w}{\partial x^2}\right)$$

$$\varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - z\left(\frac{\partial^2 w}{\partial y^2}\right)$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} - 2z\frac{\partial^2 w}{\partial x \partial y}$$

DISCUSSION:

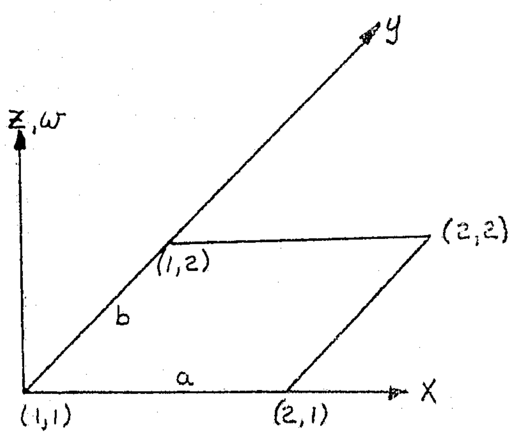
This element employs a linear stress-strain relation with all combinations of displacement gradients retained in the strain energy. A Lagrangian derivation is employed with a numerical solution obtained by direct minimization of the total potential energy.

u, v, w, w_x, w_y are all continuous at nodes. Since w_x is cubic in y on an x = constant edge, the specification of w_{xy} cannot control w_x over the entire edge, so that the slopes are not everywhere continuous (e.g., middle of a side).

Specification of continuity u_x, u_y, v_x, v_y, u_{xy}, v_{xy}, and w_{xy} really constitute a stress or strain constraint, which is undesirable, and the element could be referred to as "over-constrained", for this reason.

ELEMENT ID: P-2a

ELEMENT TYPE: Rectangular Plate



REFERENCE: [38], [39], [40]

VARIATIONS FROM BASIC ELEMENT: P-2

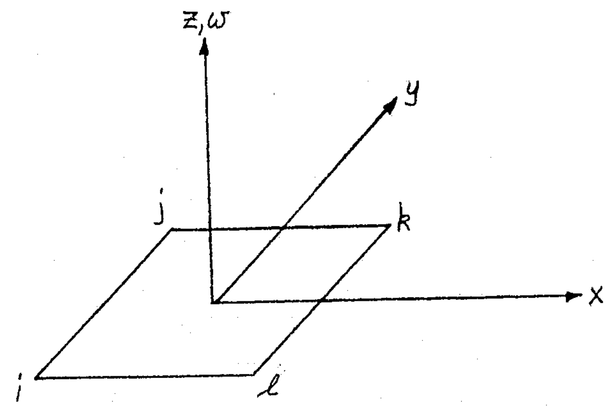
This element is a rectangular plate in which, for Ref. [38], complete generality is maintained in plate material properties so that anistropy or isotropy is considered, as well as plastic yielding prior to buckling. It has 4 nodes and 4 DOF (w, w_x, w_y, w_{xy}) per node. Only bending deformation states are considered since the analysis is limited to buckling solutions. The transverse displacement function is that of Bogner, Fox, Schmit, Ref. [25]. The conventional initial stress stiffness matrix is developed where only quadratic terms are retained in the strain energy. In addition, the standard beam element B-2 is used in conjunction with the plate to perform stiffened plate buckling analyses.

In Ref. [39], the initial stress stiffness matrix for the BFS element, Ref. [25] is suggested.

In Ref. [40] the initial stress stiffness matrix for the BFS element is developed and applied to the plastic buckling analysis of plates.

ELEMENT ID: P-3

ELEMENT TYPE: Rectangular Plate



ASSUMED DISPLACEMENT SHAPE:

$$
\begin{aligned}
32w = {} & x_m y_m \{2(x_m y_m - x_p y_p) w_1 - 4 x_p x_m (a w_{1,x} + w_1) - 4 y_p y_m (b w_{1,y} + w_1)\} + \\
& + x_m y_p \{2(x_m y_p - x_p y_m) w_2 + 4 x_p x_m (a w_{2,x} + w_2) - 4 y_p y_m (b w_{2,y} - w_2)\} + x_p y_p \{2(x_p y_p - \\
& y_m x_m) w_3 + 4 x_p x_m (a w_{3,x} - w_3) + 4 y_p y_m (b w_{3,y} - w_3)\} + x_p y_m \{2(x_p y_m - x_m y_p) w_4 - \\
& - 4 x_p x_m (a w_{4,x} - w_4) + 4 y_p y_m (b w_{4,y} + w_4)\} \quad \text{where: } x_p = \frac{x+a}{a},\ x_m = \frac{x-a}{a},\ y_p = \frac{y+b}{b},\ y_m = \frac{y-b}{b}
\end{aligned}
$$

DESCRIPTION:

This element is a rectangular plate with 4 nodes and 3 DOF (w, w_x, w_y) per node. The linear stiffness matrix was first presented by Melosh. The matrices are derived for a buckling analysis consequently only bending deformation states are required.

REFERENCE: [17], [36]

VARIATION OF THIS ELEMENT:

ADVANTAGES OR DISADVANTAGES:

ELEMENT ID: P-3

STRAIN DISPLACEMENT EQUATIONS:

(Not explicitly given)

$$\varepsilon_x = \varepsilon_{x_o} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - z\left(\frac{\partial^2 w}{\partial x^2}\right)$$

$$\varepsilon_y = \varepsilon_{y_o} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - z\left(\frac{\partial^2 w}{\partial y^2}\right)$$

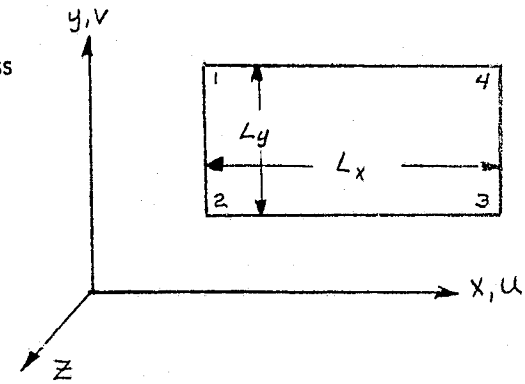
$$\gamma_{xy} = \gamma_{xyo} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} - 2z\left(\frac{\partial^2 w}{\partial x \partial y}\right)$$

DISCUSSION:

This element employs a linear stress-strain relation with only quadratic combinations of displacement gradients retained in the potential energy. A Lagrangian derivation is employed with an eiganvalue analysis performed to find the buckling loads.

ELEMENT ID: P-4

ELEMENT TYPE: Rectangular Plate, Plane Stress



ASSUMED DISPLACEMENT SHAPE:

The author speaks of an assumed linear stress distribution for σ_x and σ_y, and a constant shear stress distribution. A formal derivation was not presented.

DESCRIPTION:

This element is an isotropic rectangular plate with 4 nodes and 2 DOF (u, v) per node. It accounts for material nonlinearities but does not consider geometric nonlinearities. Only membrane deformation states are required.

REFERENCE: [19]

VARIATION OF THIS ELEMENT:

ADVANTAGES OR DISADVANTAGES:

ELEMENT ID: P-4

STRAIN DISPLACEMENT EQUATIONS: Given in general as:

$$\epsilon_x = \epsilon_{xe} + \alpha T + \epsilon_{xpt}$$
$$\epsilon_y = \epsilon_{ye} + \alpha T + \epsilon_{ypt}$$
$$\epsilon_z = \epsilon_{ze} + \alpha T + \epsilon_{zpt}$$

$$\gamma_{xy} = \gamma_{xye} + \gamma_{xypt}$$
$$\gamma_{yz} = \gamma_{yze} + \gamma_{yzpt}$$
$$\gamma_{zx} = \gamma_{zxe} + \gamma_{zxpt}$$

$$\epsilon_{xe} = \partial u / \partial x$$
$$\epsilon_{ye} = \partial v / \partial y$$
$$\epsilon_{ze} = \partial \omega / \partial z$$

$$\gamma_{xy} = \partial u / \partial y + \partial v / \partial x$$
$$\gamma_{yz} = \partial v / \partial z + \partial \omega / \partial y$$
$$\gamma_{zx} = \partial \omega / \partial x + \partial u / \partial z$$

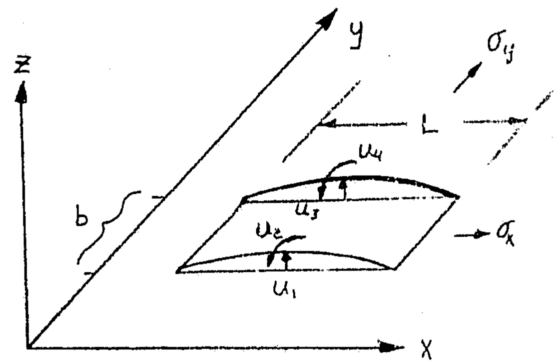
ϵ_{pt} is the total inelastic strain component

DISCUSSION:

This derivation considers material nonlinearities for a membrane plate. Plastic strains are converted to a column matrix of plastic loads. An incremental solution procedure is employed.

ELEMENT ID: P-5

ELEMENT TYPE: Strip Plate Element



ASSUMED DISPLACEMENT SHAPE:

$$w = \sin\frac{\pi x}{L}\left\lfloor(1-3\eta^2+2\eta^3),(\eta-2\eta^2+\eta^3)b,(3\eta^2-2\eta^3),(-\eta^2+\eta^3)b\right\rfloor\begin{Bmatrix}u_1\\u_2\\u_3\\u_4\end{Bmatrix}$$

$$\eta = \frac{y}{b}$$

DESCRIPTION:

This element is a rectangular strip with two line nodes and two DOF (translation, rotation) per node. A sinusoidal distribution for w is assumed along the length. Only bending deformation states are accounted for since the element is used in an eiganvalue buckling analysis.

REFERENCE: [22], [23]

VARIATION OF THIS ELEMENT:

ADVANTAGES OR DISADVANTAGES:

ELEMENT ID: P-5

STRAIN DISPLACEMENT EQUATIONS:

$$e_{xx} = -z \frac{\partial^2 w}{\partial x^2}$$

$$e_{yy} = -z \frac{\partial^2 w}{\partial y^2}$$

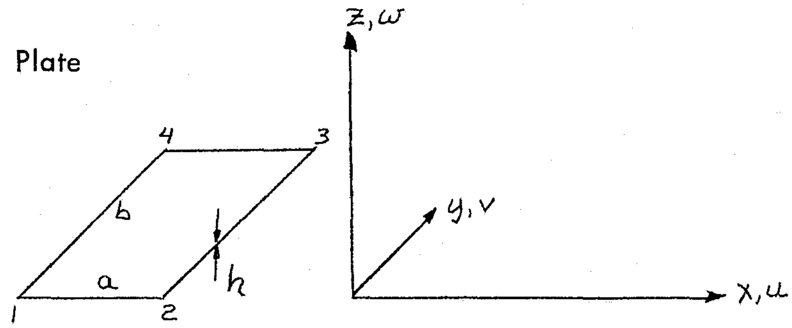
$$e_{xy} = -2z \frac{\partial^2 w}{\partial x \partial y}$$

DISCUSSION:

This element employs a linear stress-strain relation. Only quadratic products of the displacement gradients are retained in the strain energy A Lagrangian derivation is employed with the solution for buckling loads obtained by an eiganvalue analysis.

ELEMENT ID: P-6

ELEMENT TYPE: Rectangular Plate



ASSUMED DISPLACEMENT SHAPE:

$$u = \sum_{i=1}^{2} \sum_{j=1}^{2} H_{0i}^{(0)}(x)\, H_{0j}^{(0)}(y)\, u_{ij}\,; \quad H_{01}^{(0)} = -\frac{1}{a}(x-a)\,, \quad H_{02}^{(0)} = \frac{1}{a}x\,;$$ similarly for v

$$w = \sum_{i=1}^{2} \sum_{j=1}^{2} \Big[H_{0i}^{(1)}(x)\, H_{0j}^{(1)}(y)\, w_{ij} + H_{1i}^{(1)}(x)\, H_{0j}^{(1)}(y)\, w_{x\,ij} + H_{0i}^{(1)}(x)\, H_{1j}^{(1)}(y)\, w_{y\,ij} + H_{1i}^{(1)}(x)\, H_{1j}^{(1)}(y)\, w_{xy\,ij}$$

$$H_{01}^{(1)}(x) = \frac{1}{a^3}\left(2x^3 - 3ax^2 + a^3\right), \quad H_{02}^{(1)}(x) = -\frac{1}{a^3}\left(2x^3 - 3ax^2\right), \quad H_{11}^{(1)}(x) = \frac{1}{a^2}\left(x^3 - 2ax^2 + a^2 x\right)$$

$$H_{12}^{(1)}(x) = \frac{1}{a^2}\left(x^3 - ax^2\right)$$ Similarly for y direction replace x by y, and a by b.

DESCRIPTION:

This element is an isotropic rectangular plate with initial imperfections. It has 4 nodes and 6 DOF (u, v, w, w_x, w_y, w_{xy}) per node. Stretching and bending deformation states are accounted for. Displacements are approximated by Lagrangian and Hermitian interpolation polynomials. Only geometric nonlinearities are considered.

REFERENCE: [24]; see [25] for linear element.

VARIATION OF THIS ELEMENT:

ADVANTAGES OR DISADVANTAGES:

Assumed displacement state is complete and therefore converges monotonically.

Linearity of u, v probably means not suitable for large deflection-post-buckling type of problem.

ELEMENT ID: P-6

STRAIN DISPLACEMENT EQUATIONS:

$$\epsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\frac{\partial(w+w_o)}{\partial x}\right]^2 - \frac{1}{2}\left(\frac{\partial w_o}{\partial x}\right) - z\left(\frac{\partial^2 w}{\partial x^2}\right)$$

$$\epsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left[\frac{\partial(w+w_o)}{\partial y}\right]^2 - \frac{1}{2}\left(\frac{\partial w_o}{\partial y}\right)^2 - z\left(\frac{\partial^2 w}{\partial y^2}\right)$$

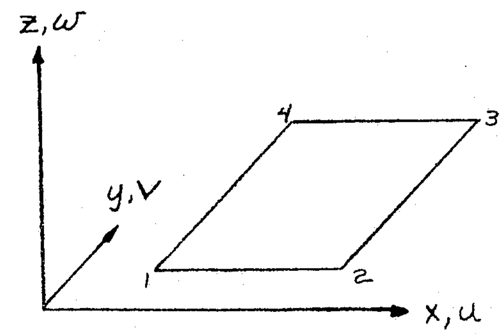
$$\epsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial(w+w_o)}{\partial x}\,\frac{\partial(w+w_o)}{\partial y} - \frac{\partial w_o}{\partial x}\,\frac{\partial w_o}{\partial y} - 2z\frac{\partial^2 w}{\partial x \partial y}$$

DISCUSSION:

This element employs a linear stress-strain relation with all combinations of displacement gradients retained in the strain energy. Initial imperfections (w_o) are included. A Lagrangian derivation is employed. The equations for both the direct iterative analysis and the step-by-step incremental procedure are presented.

ELEMENT ID: P-7

ELEMENT TYPE: Rectangular plate or shallow shell



ASSUMED DISPLACEMENT SHAPE:

$$u = \lfloor 1, x, y, xy \rfloor \{\alpha_1\}$$

$$v = \lfloor 1, x, y, xy \rfloor \{\alpha_2\}$$

$$w = \lfloor 1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, x^3y, xy^3 \rfloor \{\alpha_3\}$$

DESCRIPTION:

This element is a rectangular plate or a shallow shell. It has 4 nodes and 5 DOF (u, v, w, w_x, w_y) per node. Stretching and bending deformation states are accounted for. Only geometric nonlinearities are considered.

REFERENCE: [26]

VARIATION OF THIS ELEMENT: [10], [45]

ADVANTAGES OR DISADVANTAGES:

It does not satisfy normal slope compatibility.

ELEMENT ID: P-7

STRAIN DISPLACEMENT EQUATIONS:

$$e_1 = u_{1,x} - Z_{,xx}\,w + \tfrac{1}{2} w_{,x}^2 - \zeta w_{,xx}$$

$$e_2 = u_{2,y} - Z_{,yy}\,w + \tfrac{1}{2} w_{,y}^2 - \zeta w_{,yy}$$

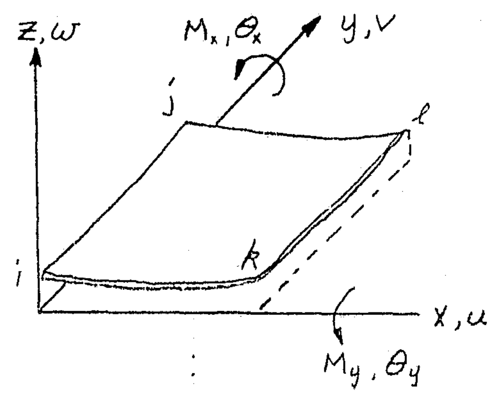
$$e_3 = u_{1,y} + u_{2,x} - 2Z_{,xy}\,w + w_{,x} w_{,y} - 2\zeta w_{,xy}$$

DISCUSSION:

This element employs a linear stress-strain relation with all combinations of displacement gradients retained in the strain energy. A Lagrangian derivation is employed with the numerical solution obtained by Newton-Raphson iteration. The formulation is also specialized to obtain the linearized incremental equations and the equations for a linear buckling analysis. Initial strains are considered.

ELEMENT ID: P-7a

ELEMENT TYPE: Rectangular Plate or Shallow Shell



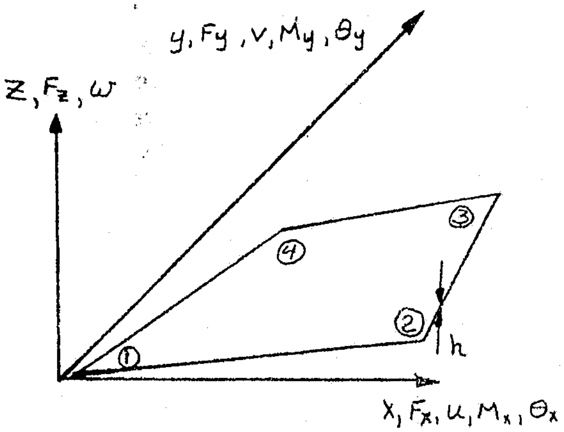
REFERENCE: [10]

VARIATIONS FROM BASIC ELEMENT: P-7

This element is an isotropic rectangular plate with initial imperfections or a shallow shell. The displacement shapes are the same as element P-7, but only quadratic products of displacement gradients are retained in the strain energy. The integrals are evaluated numerically using Gaussian quadrature formulae. The solution is obtained using mid-increment stiffness matrices. Only geometric nonlinearities are considered.

ELEMENT ID: P-7b

ELEMENT TYPE: Quadrilateral Plate



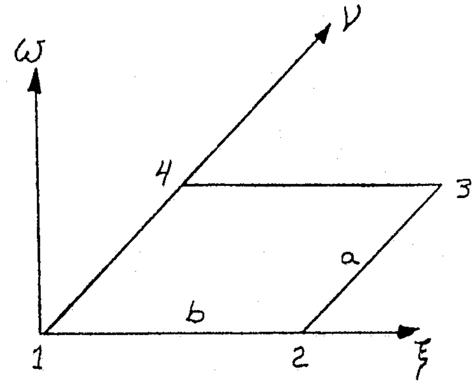
REFERENCE: [45]

VARIATIONS FROM BASIC ELEMENT: P-7

This element uses the membrane stiffness matrix of Ref. [19], (see P-4), with the 12 term polynomial of P-7 for the lateral displacement w. A linear stress-strain relation is employed with only quadratic products of displacement gradients retained in the strain energy. Several assumed transverse displacement functions are used in deriving the k' matrix including 1) the 12 term polynomial used in developing the k^o matrix, 2) a six term assumption, and 3) a four term assumption. A Lagrangian derivation is employed with the numerical solution obtained by a combined incremental step and eiganvalue analysis to determine the buckling load.

ELEMENT ID: P–8

ELEMENT TYPE: Rectangular Plate



ASSUMED DISPLACEMENT SHAPE:

$$\omega = \sum_{k=1}^{4} \sum_{\ell=1}^{3} u_{k\ell} f_{k\ell}(x,y), \quad u_{i1} = \omega_i, \quad u_{i2} = a\alpha_i, \quad u_{i3} = b\beta_i$$

$$\begin{array}{llll}
f_{11} = \phi_1(\xi)\phi_1(\nu) & f_{21} = \phi_2(\xi)\phi_1(\nu) & f_{31} = \phi_2(\xi)\phi_2(\nu) & f_{41} = \phi_1(\xi)\phi_2(\nu) \\
f_{12} = \phi_1(\xi)\phi_3(\nu) & f_{22} = \phi_2(\xi)\phi_3(\nu) & f_{32} = \phi_2(\xi)\phi_4(\nu) & f_{42} = \phi_1(\xi)\phi_4(\nu) \\
f_{13} = -\phi_3(\xi)\phi_1(\nu) & f_{23} = -\phi_4(\xi)\phi_1(\nu) & f_{33} = -\phi_4(\xi)\phi_2(\nu) & f_{43} = -\phi_3(\xi)\phi_2(\nu)
\end{array}$$

$$\phi_1(\nu) = 1 - 3\nu^2 + 2\nu^3, \quad \phi_2(\nu) = 3\nu^2 - 2\nu^3, \quad \phi_3(\nu) = \nu - 2\nu^2 + \nu^3, \quad \phi_4(\nu) = -\nu^2 + \nu^3$$

DESCRIPTION:

This element is an isotropic rectangular plate with 4 nodes and 3 DOF (w, α, β) per node. Only bending deformation states are accounted for. An initial state of constant stress is assumed and the resulting initial stress stiffness matrix is suitable for a buckling analysis.

REFERENCE: [27], [28]

VARIATION OF THIS ELEMENT: [29]

ADVANTAGES OR DISADVANTAGES:

This element omits the deformation state corresponding to constant twist and although it is fully compatible and will converge monotonically, it may not converge to the true solution.

ELEMENT ID: P-8

~~STRAIN DISPLACEMENT EQUATIONS:~~
ENERGY EQUATION:

$$U = \frac{D}{2} \iint_A \left\{ \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1-\nu) \left[\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} dx\,dy$$

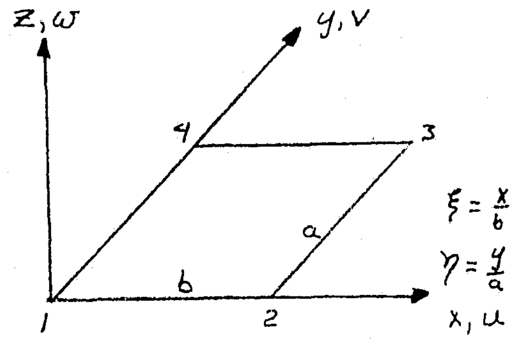
$$+ \frac{1}{2} \iint_A \left[N_x \left(\frac{\partial w}{\partial x} \right)^2 + N_y \left(\frac{\partial w}{\partial y} \right)^2 + 2 N_{xy} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right] dx\,dy$$

DISCUSSION:

This is the Papenfuss element [28], extended to a buckling analysis. It employs a linear stress-strain relation with only quadratic combinations of displacement gradients retained in the work expression. A Lagrangian derivation is employed with an eiganvalue analysis performed to find the buckling loads.

ELEMENT ID: P-8a

ELEMENT TYPE: Rectangular Plate



REFERENCE: [29]

VARIATIONS FROM BASIC ELEMENT: P-8

This is the element of Reference [28] extended for use in the large deflection of plates. A membrane displacement shape is assumed

$$u = (1-\xi)(1-\eta)u_1 + \xi(1-\eta)u_2 + \xi\eta u_3 + (1-\xi)\eta u_4 = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$$

$$v = (1-\xi)(1-\eta)v_1 + \xi(1-\eta)v_2 + \xi\eta v_3 + (1-\xi)\eta v_4 = \alpha_5 + \alpha_6 x + \alpha_7 y + \alpha_8 xy$$

and used in conjunction with the bending shape of P-8 in the strain displacement equations

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - z\,\frac{\partial^2 w}{\partial x^2}$$

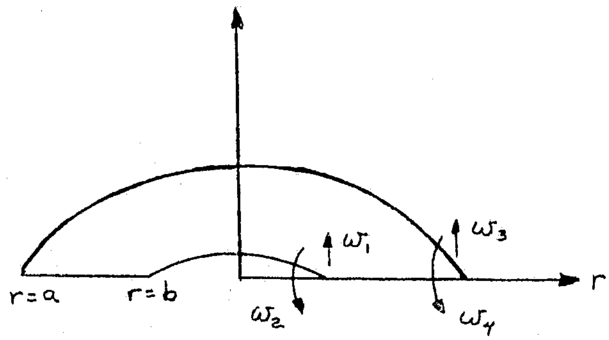
$$\varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - z\,\frac{\partial^2 w}{\partial y^2}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\,\frac{\partial w}{\partial y} - 2z\,\frac{\partial^2 w}{\partial x\,\partial y}$$

The resulting element has 4 nodes and 5 DOF (u, v, w, α, β) per node. A linear stress-strain relation is used and all combinations of displacement gradients are retained in the strain energy. The third and fourth order combinations of displacement gradients are converted to equivalent load columns and the resulting equations solved by an iterative technique. A Lagrangian derivation is employed.

ELEMENT ID: P-9

ELEMENT TYPE: Annular Plate



ASSUMED DISPLACEMENT SHAPE:

$$\omega(r) = \sum_{i=1}^{4} \phi_i(r)\omega_i, \quad \phi_i(r) = \sum_{j=1}^{4} \gamma_{ij}\rho_j(r), \quad \rho_1 = 1, \; \rho_2 = \left(\frac{r}{b}\right)^2, \; \rho_3 = \ln\left(\frac{r}{b}\right), \; \rho_4 = \left(\frac{r}{b}\right)^2 \ln\left(\frac{r}{b}\right)$$

$$\gamma_{11} = \alpha^2(\alpha^2 - 1 + 2\ln\alpha - 4\ln^2\alpha)/\Delta, \quad \gamma_{12} = (1 - 2\alpha^2\ln\alpha - \alpha^2)/\Delta, \quad \gamma_{13} = 4\alpha^2\ln\alpha/\Delta, \quad \gamma_{14} = 2(\alpha^2 - 1)/\Delta$$

$$\gamma_{21} = 2b\alpha^2\ln\alpha/\Delta, \quad \gamma_{22} = -\gamma_{12}, \quad \gamma_{23} = b\alpha^2(\alpha^2 - 1 - 2\ln\alpha)/\Delta, \quad \gamma_{24} = b(2\alpha^2\ln\alpha - \alpha^2 + 1)/\Delta$$

$$\gamma_{31} = \gamma_{12}, \quad \gamma_{32} = -\gamma_{12}, \quad \gamma_{33} = -\gamma_{13}, \quad \gamma_{34} = -\gamma_{14}, \quad \alpha = a/b, \quad \Delta = (\alpha^2 - 1)^2 - 4\alpha^2\ln^2\alpha$$

$$\gamma_{41} = b\alpha\ln\alpha(\alpha^2 - 1)/\Delta, \quad \gamma_{42} = -\gamma_{41}, \quad \gamma_{43} = b\alpha(2\alpha^2\ln\alpha - \alpha^2 + 1)/\Delta, \quad \gamma_{44} = b\alpha(\alpha^2 - 1 - 2\ln\alpha)/\Delta$$

DESCRIPTION:

This element is an isotropic annular plate with 2 ring nodes and 2 DOF (translation, rotation) per node. The displacement function is the exact solution to the circular plate equation–bending only. An initial state of stress is assumed and the resulting initial stress stiffness matrix is suitable for a buckling analysis.

REFERENCE: [30]

VARIATION OF THIS ELEMENT:

ADVANTAGES OR DISADVANTAGES:

ELEMENT ID: P-9

~~STRAIN DISPLACEMENT EQUATIONS:~~
Potential Energy:

$$V = \frac{1}{2}\int_a^b 2\pi \left\{ N_R (\omega')^2 + D\left[\left(\omega'' + \frac{\omega'}{r} \right)^2 - 2(1-\mu)\,\omega'' \frac{\omega'}{r} \right] \right\} r\,dr$$

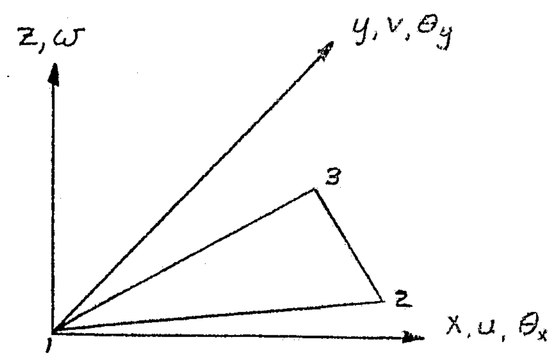
$$N_R = -N_o \frac{a^2}{a^2 - b^2}\left(1 - \frac{b^2}{r^2}\right)$$

DISCUSSION:

This element employs an isotropic, linear stress-strain relation although orthotropicity is discussed. Only quadratic combinations of displacement gradients are retained in the potential energy. A Lagrangian derivation is employed with an eiganvalue analysis performed to find the buckling loads. A circular or closure element is derived for a static analysis.

ELEMENT ID: P-10

ELEMENT TYPE: Triangular Plate



ASSUMED DISPLACEMENT SHAPE:

Never explicity given. Stated as being:

Membrane displacement functions are those of constrant strain triangle;

Bending displacement functions are those of the Hsich-Clough-Tocher triangle.

DESCRIPTION:

This element is an isotropic triangular plate with 3 nodes and 5 DOF ($u, v, w, \theta_x, \theta_y$) per node. Membrane and bending deformation states are included in the derivation. This is an extension of the HCT triangle to postbuckling analyses. Only geometric nonlinearities are considered.

REFERENCE: [31], [32]

VARIATION OF THIS ELEMENT:

ADVANTAGES OR DISADVANTAGES:

ELEMENT ID: P-10

STRAIN DISPLACEMENT EQUATIONS:

Strain increment expression

$$\Delta E_{ij} = \frac{1}{2}\left\{ u_{i,j} + u_{j,i} + u_{k,i}u_{k,j} + \bar{u}_{k,i}u_{k,j} + u_{k,i}\bar{u}_{k,j} \right\}$$

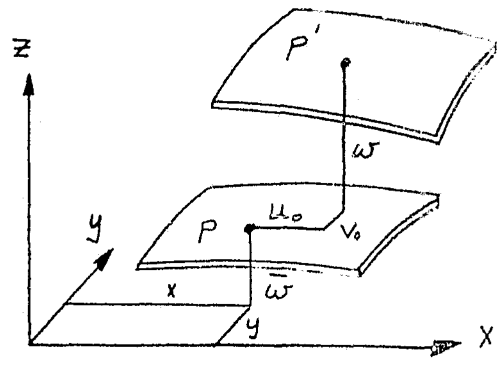
$\bar{u}_i$ - displacements in deformed equilibrium (initial) configuration

$\bar{u}_i + u_i$ - displacements in deformed equilibrium (incremented) configuration

DISCUSSION:

This element employs a linear stress-strain relation with only quadratic combinations of displacement gradients in the expression defining the principle virtual displacements. Convected coordinates are used to form the incremental equations of equilibrium and the resulting solutions are related to Lagrangian quantities by transformation. The solution procedure is iterative as well as incremental.

ELEMENT ID: P-11



ELEMENT TYPE:

Quadrilateral plate or
Shallow Shell

ASSUMED DISPLACEMENT SHAPE:

Never explicitly given. Described as

in-plane: Zienkiewicz-Irons isoparametric quadrilateral [33]

bending: the Q-19 quadrilateral plate of ref. [34]

DESCRIPTION:

This element is an isotropic quadrilateral plate with 4 nodes and 5 DOF ($u, v, w, \theta_x, \theta_y$) per node. Membrane and bending deformation states are included in the derivation. This is an extension of the Q-19 plate to the geometrically nonlinear regime.

REFERENCE: [35]

VARIATION OF THIS ELEMENT:

ADVANTAGES OR DISADVANTAGES:

ELEMENT ID: P-11

STRAIN DISPLACEMENT EQUATIONS: (Marguerre Shallow Shell)

$$\varepsilon_x = \frac{\partial u_0}{\partial x} + \frac{\partial \bar{w}}{\partial x}\frac{\partial w}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - \zeta\frac{\partial^2 w}{\partial x^2}$$

$$\varepsilon_y = \frac{\partial v_0}{\partial y} + \frac{\partial \bar{w}}{\partial y}\frac{\partial w}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - \zeta\frac{\partial^2 w}{\partial y^2}$$

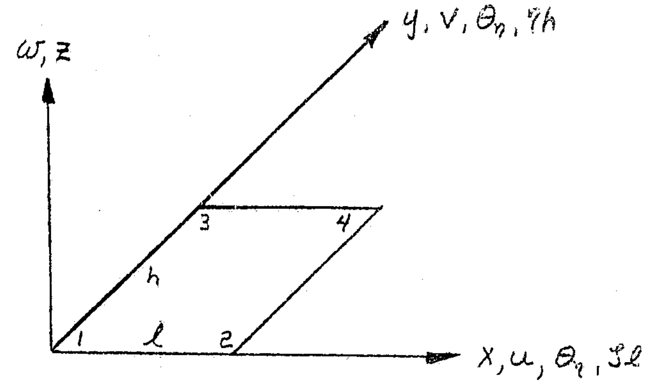
$$\gamma_{xy} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial \bar{w}}{\partial x}\frac{\partial w}{\partial y} + \frac{\partial \bar{w}}{\partial y}\frac{\partial w}{\partial x} - 2\zeta\frac{\partial^2 w}{\partial x\,\partial y}$$

DISCUSSION:

This element employs a linear stress-strain relation with all combinations of displacement gradients retained in the strain energy. A Lagrangian derivation is employed to obtain the total load-deflection equation which is nonsymmetric and an incremental relation. The nonsymmetric stiffness matrix does not cause a problem because of the combined incremental and iterative solution procedure.

ELEMENT ID: P-12

ELEMENT TYPE: Rectangular Plate



ASSUMED DISPLACEMENT SHAPE:

$$\omega(x,y)=\sum_{i=1}^{2}\sum_{j=1}^{2}\left[H_{oi}^{(1)}(x)H_{oj}^{(1)}(y)\omega_{ij}+H_{1i}^{(1)}(x)H_{oj}^{(1)}(y)\omega_{xij}+H_{oi}^{(1)}(x)H_{1j}^{(1)}(y)\omega_{yij}+H_{1i}^{(1)}(x)H_{oj}^{(1)}(y)\omega_{xyij}\right]$$

$$u(x,y)=\sum_{i=1}^{2}\sum_{j=1}^{2}H_{oi}^{(0)}(x)H_{oj}^{(0)}(y)u_{ij} \quad \text{similarly for } v(x,y)$$

$$H_{01}^{(1)}(x)=(2x^3-3\ell x^2+\ell^3)/\ell^3,\ H_{11}^{(1)}(x)=(x^3-2\ell x^2+\ell^2 x)/\ell^2,\ H_{02}^{(1)}(x)=-(2x^3-3\ell x^2)/\ell^3$$

$$H_{12}^{(1)}(x)=(x^3-\ell x^2)/\ell^2,\ H_{01}^{(0)}(x)=-(x-1)/\ell,\ H_{02}^{(0)}(x)=x/\ell$$

$$\text{where } \omega_x=-\theta_\eta/h,\ \omega_y=\theta_\xi/\ell,\ \omega_{xy}=\psi_{\xi\eta}/\ell h$$

DESCRIPTION:

This element is an isotropic rectangular plate with 4 nodes and 6 DOF ($u, v, w, w_x, w_y, w_{xy}$) per node. Stretching and bending deformation states are accounted for. Transverse deflections are approximated by superposition of first order Hermite polynomials and in-plane deflections approximated by zeroth order Hermite polynomials. Only geometric nonlinearities are considered.

REFERENCE:

[37]

VARIATION OF THIS ELEMENT: See also P-2, P-2a, S-5

ADVANTAGES OR DISADVANTAGES:

ELEMENT ID: P-12

STRAIN DISPLACEMENT EQUATIONS:

$$e_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - z\left(\frac{\partial^2 w}{\partial x^2}\right)$$

$$e_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - z\left(\frac{\partial^2 w}{\partial y^2}\right)$$

$$2\,e_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\,\frac{\partial w}{\partial y} - 2z\,\frac{\partial^2 w}{\partial x\,\partial y}$$

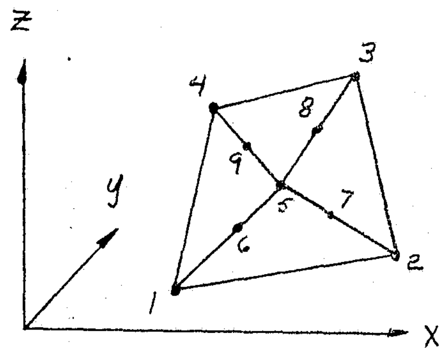
DISCUSSION:

This element employs a linear stress-strain relation with all combinations of displacement gradients retained in the strain energy. The third and fourth order matrices are evaluated numerically. A Lagrangian derivation is employed with a numerical solution with an energy perturbation approach used for the post-buckling regime. Initial imperfections are considered.

ELEMENT ID: P-13

ELEMENT TYPE:

Quadrilateral Plate Composed of Four Triangles, i.e., the Q-19 Element.



ASSUMED DISPLACEMENT SHAPE:

This derivation is carried out in triangular or natural coordinates. The displacement function is given explicitly in terms of triangular coordinates but the explanation of these coordinates is beyond the scope of this document. The bending displacement function is equivalent to a cubic in the rectangular cartesian coordinates. The initial stress stiffness matrix is developed for three displacement functions; a compatible cubic, an incompatible cubic, and a linear function.

DESCRIPTION:

This element is an anisotropic quadrilateral plate with 4 nodes and 3 DOF (w, θ_x, θ_y) per node. It is composed of 4 LCCT-11 elements (a version of the HCT element [32]) with the internal DOF reduced out on the elemental level. The matrices are derived for a buckling analysis, consequently only bending and shearing deformation states are used.

REFERENCE: [41]

VARIATION OF THIS ELEMENT:

ADVANTAGES OR DISADVANTAGES:

Fully compatible quadrilateral.

ELEMENT ID: P-13

STRAIN DISPLACEMENT EQUATIONS: They are not given explicitly but are equivalent to:

$$e_{xx} = \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - z\left(\frac{\partial^2 w}{\partial x^2}\right)$$

$$e_{yy} = \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - z\left(\frac{\partial^2 w}{\partial y^2}\right)$$

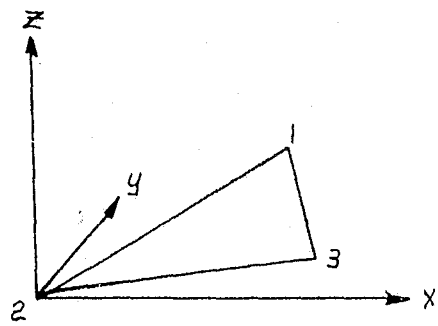
$$2e_{xy} = \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} - 2z\frac{\partial^2 w}{\partial x \partial y}$$

DISCUSSION:

This element employs a linear anisotropic stress-strain relation. Only quadratic products of displacement gradients are retained in the strain energy in order to develop the initial stress stiffness matrix. A Lagrangian derivation is employed with an eiganvalue analysis performed to find the buckling loads.

ELEMENT ID: P-14

ELEMENT TYPE: Triangular Plate



ASSUMED DISPLACEMENT SHAPE:

Not given explicitly, but stated as being the BCIZ displacement functions of Ref. [43].

DESCRIPTION:

This element is an isotropic triangular plate with 3 nodes and 3 DOF (w, θ_x, θ_y) per node. Numerical integration is used in deriving the element, permitting thickness variation within an element. The matrices are derived for a buckling analysis, thus only bending deformation states are used.

REFERENCE: [44]

VARIATION OF THIS ELEMENT:

ADVANTAGES OR DISADVANTAGES:

ELEMENT ID: P-14

STRAIN DISPLACEMENT EQUATIONS:

Not explicitly given.

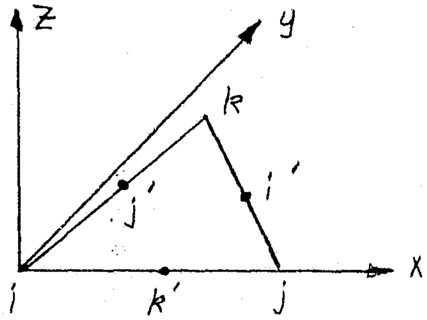
DISCUSSION:

This element employs a linear stress-strain relation in the element derivation. Only guadratic products of displacement gradients are retained in the strain energy in order to develop the initial stress stiffness matrix. In Ref. [43] the stiffness matrix is generated through the use of area coordinates. A Lagrangian derivation is employed with an eiganvalue analysis performed to find the buckling loads. Ref. [44] presents numerous results using this element.

HANDBOOK OF NONLINEAR FINITE ELEMENTS

ELEMENT ID: P-15

ELEMENT TYPE: Triangular Plate



ASSUMED DISPLACEMENT SHAPE:

$$\omega(x,y) = \{1, x, y, \ldots xy^4, y^5\} \begin{Bmatrix} a_1 \\ \vdots \\ a_{21} \end{Bmatrix}$$

DESCRIPTION:

This element is an isotropic triangular plate with 3 nodes and 6 DOF per node. The derivation requires additional DOF of normal slopes at the mid-side nodes which are eliminated by imposing a cubic variation in the slope. Only bending deformation states are presented. Material non-linearities are considered.

REFERENCE: [48]

VARIATION OF THIS ELEMENT:

ADVANTAGES OR DISADVANTAGES:

ELEMENT ID: P-15

STRAIN DISPLACEMENT EQUATIONS: Not given explicitly but refers to Ref. [2] for the initial stress stiffness contribution

$$\epsilon_x = \epsilon_{x^0} + \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2$$

$$\varepsilon_y = \varepsilon_y{}^0 + \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2$$

$$\gamma_{xy} = \gamma_{xy}{}^0 + \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\,\frac{\partial w}{\partial y}$$

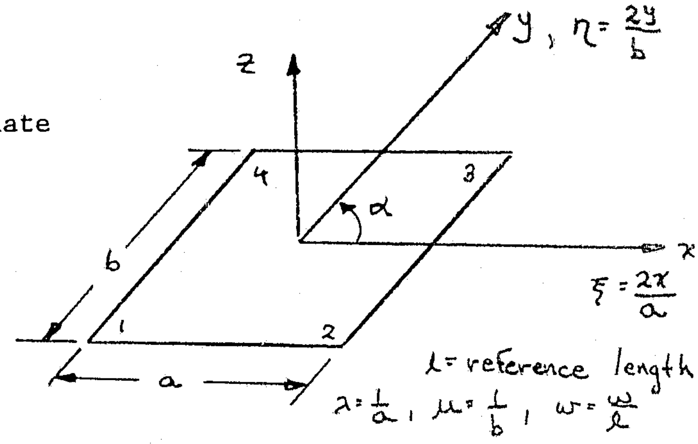
DISCUSSION:

This element considers material nonlinearities that obey the Prager-Ziegler kinematic hardening theory and therefore accounts for the Bauschinger effect for biaxial stress states. The incremental theory of plasticity is used with linear elastic and perfectly plastic conditions included as special cases. A Lagrangian derivation is employed with only quadratic products of displacement gradients retained in the strain energy. Plastic strains are treated in the same way as the equivalence between temperature gradients and body forces. This results in an "initial strain stiffness matrix" which is not symmetric. A number of solution procedures are presented including:

Displacement Method - Predictor Procedure
Strain Method - Predictor Procedure
Displacement Method - Direct Substitution Procedure
Strain Method - Direct Substitution Procedure

ELEMENT ID: P-16

ELEMENT TYPE: Parallelogram plate



ASSUMED DISPLACEMENT SHAPE:

Membrane:

1) $u/l = \xi/4$ $\quad v/l = 0$
2) $u/l = 0$ $\quad v/l = \eta/4$
3) $u/l = \xi\eta/4$ $\quad v/l = 0$
4) $u/l = 0$ $\quad v/l = 9\eta/4$
5) $u/l = \eta/8\mu$ $\quad v/l = \xi/8\lambda$
6) $u/l = 1/4$ $\quad v/l = 0$
7) $u/l = 0$ $\quad v/l = 1/4$
8) $u/l = -\eta s/8\mu$ $\quad v/l = \xi s/8\lambda$

Bending:

1) $\varpi = \frac{1}{16\lambda}(1-\xi^2)$
2) $\varpi = \frac{1}{16\mu}(1-\eta^2)$
3) $\varpi = -\frac{1}{16\lambda}\xi(1-\xi^2)$
4) $\varpi = -\frac{1}{16\mu}\eta(1-\eta^2)$
5) $\varpi = -\frac{1}{32\lambda}\eta(1-\xi^2)(3-\eta^2)$
6) $\varpi = -\frac{1}{32\mu}\xi(3-\xi^2)(1-\eta^2)$
7) $\varpi = \frac{1}{32\lambda}\xi\eta(1-\xi^2)(3-\eta^2)$
8) $\varpi = \frac{1}{32\mu}\xi\eta(3-\xi^2)(1-\eta^2)$
9) $\varpi = \frac{1}{4}\xi\eta$
10) $\varpi = \frac{1}{4}$
11) $\varpi = -\frac{1}{8\lambda}\xi$
12) $\varpi = \frac{1}{8\mu}\eta$

DESCRIPTION:

This is an anisotropic parallelogram plate with 4 nodes and 5 DOF (u,v,w,φ,θ) per node. Stretching and bending deformation states are accounted for. Both geometric and material nonlinearities are accounted for. The deviation is carried out in natural coordinates.

REFERENCE: [63]

VARIATION OF THIS ELEMENT:

ADVANTAGES OR DISADVANTAGES:

ELEMENT ID: P-16

STRAIN DISPLACEMENT EQUATIONS:

The derivation is carried out in several parts with the membrane and bending strains having the usual form

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} - z\,\frac{\partial^2 w}{\partial x^2}$$
$$\varepsilon_{yy} = \frac{\partial v}{\partial x} - z\,\frac{\partial^2 w}{\partial y^2}$$
$$\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - \sqrt{2}\,z\,\frac{\partial^2 w}{\partial x \partial y}$$

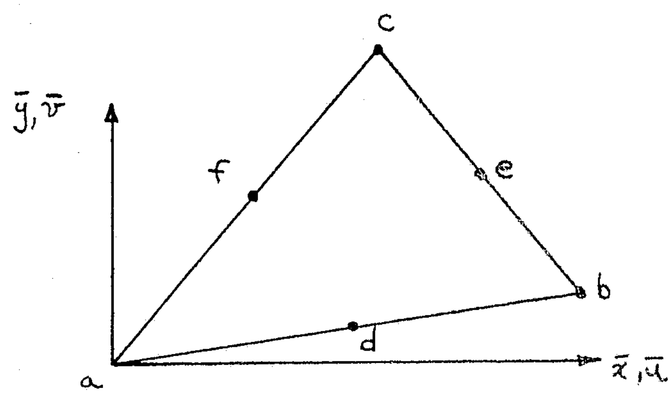
A geometrical stiffness matrix is developed for buckling analyses and the derivation proceeds from an expression for the generalized load matrix. The details of this expression are beyond the scope of this handbook. The reader is referred to page 91 of Ref. 63 for the derivation of this matrix.

DISCUSSION:

This element employs a linear stress-strain relation with only quadratic combinations of displacement gradient used to form the elemental matrices. A Lagrangian derivation is employed using oblique coordinates. An eigenvalue analysis is performed to find the buckling loads, and an incremental approach is used in large deflection problems.

ELEMENT ID: P-17

ELEMENT TYPE: Membrane Triangular Plate



ASSUMED DISPLACEMENT SHAPE:

$$\bar{u}(\bar{x},\bar{y}) = \lfloor 1 \quad \bar{x} \quad \bar{y} \quad \tfrac{1}{2}\bar{x}^2 \quad \bar{x}\bar{y} \quad \tfrac{1}{2}\bar{y}^2 \rfloor \{c_1\}$$
$$\bar{v}(\bar{x},\bar{y}) = \lfloor 1 \quad \bar{x} \quad \bar{y} \quad \tfrac{1}{2}\bar{x}^2 \quad \bar{x}\bar{y} \quad \tfrac{1}{2}\bar{y}^2 \rfloor \{c_2\}$$
$$\bar{w}(\bar{x},\bar{y}) = \lfloor 1 \quad \bar{x} \quad \bar{y} \quad \tfrac{1}{2}\bar{x}^2 \quad \bar{x}\bar{y} \quad \tfrac{1}{2}\bar{y}^2 \rfloor \{c_3\}$$

DESCRIPTION:

This element is a plane stress, linear strain geometrically orthotropic triangular plate with 6 nodes and 3 DOF (u,v,w) per node. Only the stretching deformation state is accounted for. Only geometric nonlinearities are considered.

REFERENCE:
[65], [64], [63]

VARIATION OF THIS ELEMENT:

ADVANTAGES OR DISADVANTAGES:

ELEMENT ID: P-17

STRAIN DISPLACEMENT EQUATIONS:

$$\epsilon_{\bar{x}} = \frac{\partial \bar{u}}{\partial \bar{x}} + \frac{1}{2}\left(\frac{\partial \bar{w}}{\partial \bar{x}}\right)^2$$

$$\epsilon_{\bar{y}} = \frac{\partial \bar{v}}{\partial \bar{y}} + \frac{1}{2}\left(\frac{\partial \bar{w}}{\partial \bar{y}}\right)^2$$

$$\gamma_{\bar{x}\bar{y}} = \frac{\partial \bar{u}}{\partial \bar{y}} + \frac{\partial \bar{v}}{\partial \bar{x}} + \frac{\partial \bar{w}}{\partial \bar{x}}\frac{\partial \bar{w}}{\partial \bar{y}}$$

DISCUSSION:

This element has an assumed quadratic displacement state that yields a linear strain, linear stress element. A linear stress-strain relation is assumed and only quadratic terms are retained in the strain energy. A Lagrangian derivation is employed. In Ref. [64] the strain displacement relations are modified to include small strain large rotation effects

$$\Delta\epsilon_{\bar{x}} = \frac{\partial \Delta\bar{u}}{\partial \bar{x}} + \frac{1}{2}(\Delta\omega_{\bar{z}})^2 + \frac{1}{2}\left(\frac{\partial \Delta\bar{w}}{\partial \bar{x}}\right)^2$$

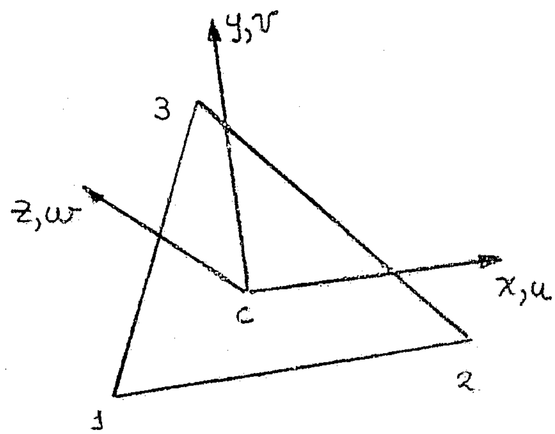
$$\Delta\epsilon_{\bar{y}} = \frac{\partial \Delta\bar{v}}{\partial \bar{y}} + \frac{1}{2}(\Delta\omega_{\bar{z}})^2 + \frac{1}{2}\left(\frac{\partial \Delta\bar{w}}{\partial \bar{y}}\right)^2$$

$$\Delta\gamma_{\bar{x}\bar{y}} = \frac{\partial \Delta\bar{u}}{\partial \bar{y}} + \frac{\partial \Delta\bar{v}}{\partial \bar{x}} + \frac{\partial \Delta\bar{w}}{\partial \bar{x}}\frac{\partial \Delta\bar{w}}{\partial \bar{y}} \quad , \quad \Delta\omega_{\bar{z}} = \frac{1}{2}\left(\frac{\partial \Delta\bar{v}}{\partial \bar{x}} - \frac{\partial \Delta\bar{u}}{\partial \bar{y}}\right)$$

The derivation includes hybrid 4 and 5 node elements as special cases. Centroidal values of the stress resultants are used and the solution employs convected coordinates (updated geometry). The stress-strain relation can be orthotropic and nonlinear material behavior can be elastic-ideally-plastic, elastic linear hardening or elastic nonlinear hardening. In Reference [63] these elements are referred to as the TRIM4 through TRIM6 family.

ELEMENT ID: P-18

ELEMENT TYPE: Triangular plate



ASSUMED DISPLACEMENT SHAPE:

$$u = a_0 + a_1 x + a_2 y + \cdots + a_{10} y^3$$

$$v = b_0 + b_1 x + b_2 y + \cdots + b_{10} y^3$$

$$w = c_0 + c_1 x + c_2 y + \cdots + c_{10} y^3 + \cdots + c_{21} y^5$$

DESCRIPTION:

This is an orthotropic triangular plate with 3 nodes and 12 DOF ($u,v,w,\theta_x,\theta_y,\theta_z,\epsilon_x,\epsilon_y,\epsilon_{xy},K_x,K_y,K_{xy}$) per node. Stretching and bending deformation states are accounted for. A linear thickness variation is allowed.

$\epsilon_x, \epsilon_y, \epsilon_{xy}$ - strains
K_x, K_y, K_{xy} - curvatures

REFERENCE: [64]

VARIATION OF THIS ELEMENT: [41]

ADVANTAGES OR DISADVANTAGES:

The inplane displacements are complete cubic polynomials and the out-of-plane displacement is a complete quintic polynomial. For planar structures displacements and rotations will be compatible.

C2

ELEMENT ID: P-18

STRAIN DISPLACEMENT EQUATIONS:

$$\epsilon_x^m = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{8}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)^2$$

$$\epsilon_y^m = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + \frac{1}{8}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)^2$$

$$\gamma_{xy}^m = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}$$

$$\epsilon_x^b = -z\frac{\partial^2 w}{\partial x^2}$$

$$\epsilon_y^b = -z\frac{\partial^2 w}{\partial y^2}$$

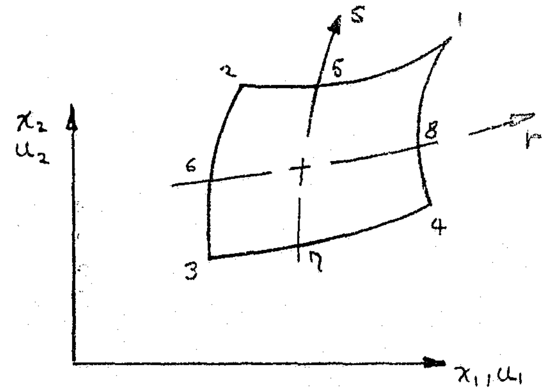
$$\gamma_{xy}^b = -2z\frac{\partial^2 w}{\partial x \partial y}$$

DISCUSSION:

This element superimposes the quadratic strain membrane triangle of Felippa with quintic displacement bending triangle of Bell. Excess degrees of freedom at the centroid - u_c, v_c are reduced by Gaussian elimination and in bending the normal slopes are required to be cubic along an edge. The element employs a linear elastic stress-strain relation with only quadratic products of displacement gradients retained in the strain energy. All stiffness matrix integrations are performed using fifth order Gauss-Legendre integration formulas. The geometric stiffness matrix was developed to be used with an updated coordinate system approach. Plasticity is accounted for using Hill's yield criterion and the kinematic hardening proposed by Prager and Ziegler.

ELEMENT ID: P-19

ELEMENT TYPE: Two dimensional plane stress, plane strain, or axisymmetric ring element.



ASSUMED DISPLACEMENT SHAPE:

$$u_1 = \sum_{k=1}^{N} h_k u_1^k \,, \quad u_2 = \sum_{k=1}^{N} h_k u_2^k$$

where the interpolation functions h_k are given on the next page

Node	r	s
1	1	1
2	-1	1
3	-1	-1
4	1	-1
5	0	1
6	-1	0
7	0	-1
8	1	0

DESCRIPTION:

This is 3 to 8 node isoparametric element with 2 DOF (u_1, v_1) per node. It can be used as an orthotropic plane stress, plane strain or axisymmetric ring element. Only stretching deformation states are included but both geometric and material nonlinearities are accounted for.

REFERENCE: [69]

VARIATION OF THIS ELEMENT:

ADVANTAGES OR DISADVANTAGES:

ELEMENT ID: P-19

DISPLACEMENT INTERPOLATION FUNCTION

Define

$$R = 1+r$$
$$S = 1+s$$
$$\bar{R} = 1-r$$
$$\bar{S} = 1-s$$
$$R^* = 1-r^2$$
$$S^* = 1-s^2$$

	Delete if node I is not included			
	I = 5	I = 6	I = 7	I = 8
$h_1 = \frac{1}{4} RS$	$-(\frac{1}{2})h_5$			$-(\frac{1}{2})h_8$
$h_2 = \frac{1}{4} \bar{R}S$	$-(\frac{1}{2})h_5$	$-(\frac{1}{2})h_6$		
$h_3 = \frac{1}{4} \overline{RS}$		$-(\frac{1}{2})h_6$	$-(\frac{1}{2})h_7$	
$h_4 = \frac{1}{4} R\bar{S}$			$-(\frac{1}{2})h_7$	$-(\frac{1}{2})h_8$
$h_5 = \frac{1}{2} R^*S$				
$h_6 = \frac{1}{2} \bar{R}S^*$				
$h_7 = \frac{1}{2} R^*\bar{S}$				
$h_8 = \frac{1}{2} RS^*$				

ELEMENT ID: P-19

STRAIN DISPLACEMENT EQUATIONS:

Both total Langrangian and updated Lagrangian formulations are given on the following page.

DISCUSSION:

This is an isoparametric quadrilateral element that can have from 3 to 8 nodes and thus, model triangular and quadrilateral planforms plus refined elements to be used in the transition from coarse to fine grids. The element can be used in a linear elastic analysis or a nonlinear material analysis. For geometric nonlinearities, both total Lagrangian and updated Lagrangian formulations are presented.

P-19

STRAIN DISPLACEMENT EQS. (Continued)

A. TOTAL LAGRANGIAN FORMULATION

1. Incremental Strains

$$ {}_0\varepsilon_{11} = {}_0u_{1,1} + {}^t_0u_{1,1}\,{}_0u_{1,1} + {}^t_0u_{2,1}\,{}_0u_{2,1} + \tfrac{1}{2}[({}_0u_{1,1})^2 + ({}_0u_{2,1})^2] $$

$$ {}_0\varepsilon_{22} = {}_0u_{2,2} + {}^t_0u_{1,2}\,{}_0u_{1,2} + {}^t_0u_{2,2}\,{}_0u_{2,2} + \tfrac{1}{2}[({}_0u_{1,2})^2 + ({}_0u_{2,2})^2] $$

$$ {}_0\varepsilon_{12} = \tfrac{1}{2}[{}_0u_{1,2} + {}_0u_{2,1}] + \tfrac{1}{2}[{}^t_0u_{1,1}\,{}_0u_{1,2} + {}^t_0u_{2,1}\,{}_0u_{2,2} + {}^t_0u_{1,2}\,{}_0u_{1,1} + $$

$$ {}^t_0u_{2,2}\,{}_0u_{2,1}] + \tfrac{1}{2}[{}_0u_{1,1}\,{}_0u_{1,2} + {}_0u_{2,1}\,{}_0u_{2,2}] $$

$$ {}_0\varepsilon_{33} = \frac{u_1}{{}^0x_1} + \frac{{}^tu_1 u_1}{({}^0x_1)^2} + \frac{1}{2}\left(\frac{u_1}{{}^0x_1}\right)^2 \qquad \text{(for axisymmetric analysis)} $$

where $ {}_0u_{i,j} = \dfrac{\partial u_i}{\partial\, {}^0x_j}\ ; \quad {}^t_0u_{i,j} = \dfrac{\partial\, {}^tu_i}{\partial\, {}^0x_j} $

B. UPDATED LAGRANGIAN FORMULATION

1. Incremental Strains

$$ {}_t\varepsilon_{11} = {}_tu_{1,1} + \tfrac{1}{2}[({}_tu_{1,1})^2 + ({}_tu_{2,1})^2] $$

$$ {}_t\varepsilon_{22} = {}_tu_{2,2} + \tfrac{1}{2}[({}_tu_{1,2})^2 + ({}_tu_{2,2})^2] $$

$$ {}_t\varepsilon_{12} = \tfrac{1}{2}[{}_tu_{1,2} + {}_tu_{2,1}] + \tfrac{1}{2}[{}_tu_{1,1}\,{}_tu_{1,2} + {}_tu_{2,1}\,{}_tu_{2,2}] $$

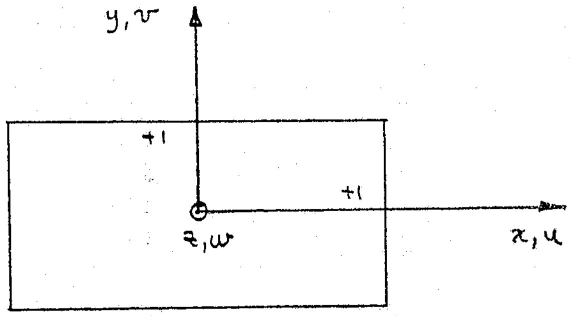
$$ {}_t\varepsilon_{33} = \frac{u_1}{{}^tx_1} + \frac{1}{2}\left(\frac{u_1}{{}^tx_1}\right)^2 \qquad \text{(for axisymmetric analysis)} $$

where $ {}_tu_{i,j} = \dfrac{\partial u_i}{\partial\, {}^tx_j} $

ELEMENT ID: P-20

ELEMENT TYPE:

Rectangular Plate



ASSUMED DISPLACEMENT SHAPE:

$$u = \beta_1 + x\beta_2 + y\beta_3 + xy\beta_3$$

$$v = \beta_5 + x\beta_6 + y\beta_7 + xy\beta_8$$

$$w = x^3\beta_9 + x^2\beta_{10} + x\beta_{11} + y^3\beta_{12} + y^2\beta_{13} + y\beta_{14} + x^3y\beta_{15} + x^2y\beta_{16} + xy\beta_{17} + xy^3\beta_{18} + xy^2\beta_{19} + \beta_{20}$$

The rectangular plate elements are normalized such that x and y range from -1 to +1.

DESCRIPTION:

This is an isotropic rectangular plate element with 4 nodes and 5 DOF ($u, v, w, w_{,x}, w_{,y}$) per node. Both geometric and material nonlinearities are accounted for in the derivation which includes the effects of stretching, shearing and bending deformation states.

REFERENCE: [87], [88]

VARIATION OF THIS ELEMENT:

ADVANTAGES OR DISADVANTAGES:

ELEMENT ID: P-20

STRAIN DISPLACEMENT EQUATIONS:

$$\gamma_{11} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - z\,\frac{\partial^2 w}{\partial x^2}$$

$$\gamma_{22} = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - z\,\frac{\partial^2 w}{\partial y^2}$$

$$\gamma_{12} = \frac{1}{2}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right] - z\,\frac{\partial^2 w}{\partial x\,\partial y}$$

DISCUSSION:

This element was developed for the transient response of a structure to blast loads. It uses isotropic materials with the von Mises yield condition, the Prandtl-Reuss normal flow rule, and isotropic work hardening. A Gaussian quadrature formula is used for spacial integration and control finite differences in the time domain.

ELEMENT ID: P-21

ELEMENT TYPE:

Flat triangular plate-shell element

ASSUMED DISPLACEMENT SHAPE:

Transverse displacement field (a) BCIZ, Ref. [43] plate shape functions with (b) Razzaque Ref. [91] derivative smoothing [(b) only recently added].

In plane: linear displacement field; rigid body motions included explicity.

DESCRIPTION:

This is a triangular plate element with 3 nodes and 6 degrees of freedom/node. Stretching and bending deformation states are included in the formulation.

REFERENCE: [89], [90]

VARIATION OF THIS ELEMENT:

ADVANTAGES OR DISADVANTAGES:

ELEMENT ID: P-21

STRAIN DISPLACEMENT EQUATIONS:

$$\hat{\varepsilon}_{ij}^{\text{midplane}} = \frac{1}{2}\left(\frac{\partial \hat{u}_i^{def}}{\partial \hat{x}_j} + \frac{\partial \hat{u}_j^{def}}{\partial \hat{x}_i} + \hat{\omega}_{ki}\hat{\omega}_{kj}\right)$$

$\hat{\varepsilon}_{ij}$ ... strain measured in connected coordinates, a rigid coordinate system X_i that rotates with the element.

$u_i^{def} = u_i - u_i^{rig}$ where u_i and u_i^{rig} are total and rigid body displacements.

$\hat{\omega}_{ki}$ ... rotation relative to convected coordinates $\hat{X}_i$

DISCUSSION:

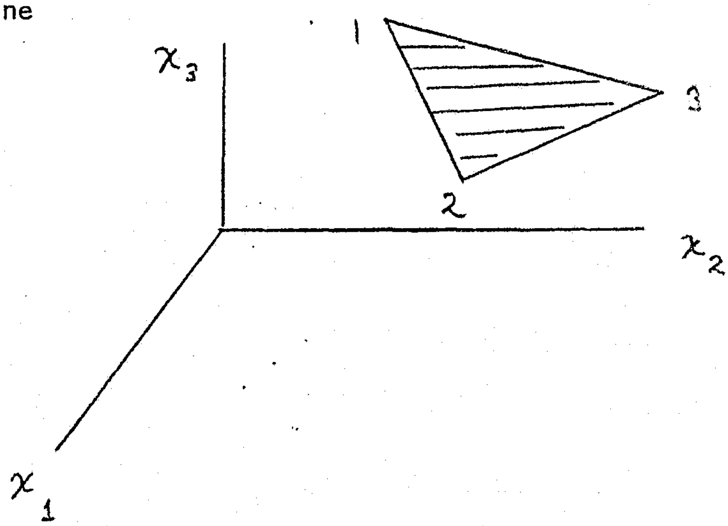
Element is applicable to arbitrarily large rotations of plates and shells with small strains and moderate variations of rotations within an element; a convected coordinate procedure is used. Plate bending shape functions are nonconforming and additional incompatibilities are introduced by matching of linear in-plane and cubic transverse deflections; however, no difficulties have been detected in about 20 sample problems.

Second order terms in strain displacement equations are automatically omitted whenever variations of rotations within an element is small. Element is computationally very efficient and quite accurate because convected coordinate formulation implicity includes all rigid body modes.

Element has been used with elastic-plastic (perfectly plastic and linear, isotropic work hardening) stress-strain law in a transient analysis program with explicity temporal integration and (unreported) a static program.

ELEMENT ID: P-22

ELEMENT TYPE: Triangular membrane



ASSUMED DISPLACEMENT SHAPE:

Linear

DESCRIPTION:

No. of nodes : 3
Degree of freedom: 9
Coordinate system: Cartesian
Isotropic materials

REFERENCE: [114], [113], [107], [115], [117], [118].

VARIATION OF THIS ELEMENT: [1], [2], [21], [60], [94].

ADVANTAGES OR DISADVANTAGES:

Advantages or disadvantages:

Advantages: Computational simplicity

Strain Displacement Equations:

$$\gamma_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{m,i}u_{m,j})$$

Discussion:

a. Stress-Strain equations: $\sigma^{ij} = \frac{1}{2}(\frac{\partial W}{\partial \gamma_{ij}} + \frac{\partial W}{\partial \gamma_{ji}}) + pG^{ij}$

b. Terms retained in the strain energy: All

c. Derivations from other than strain energy considerations: Galerkin Methods, Conservation of Energy

d. Unique derivation techniques: See (c) above.

e. Lagrangian, Eulerian, convected: Lagrangian (convected on material)

f. Numerical integration schemes:

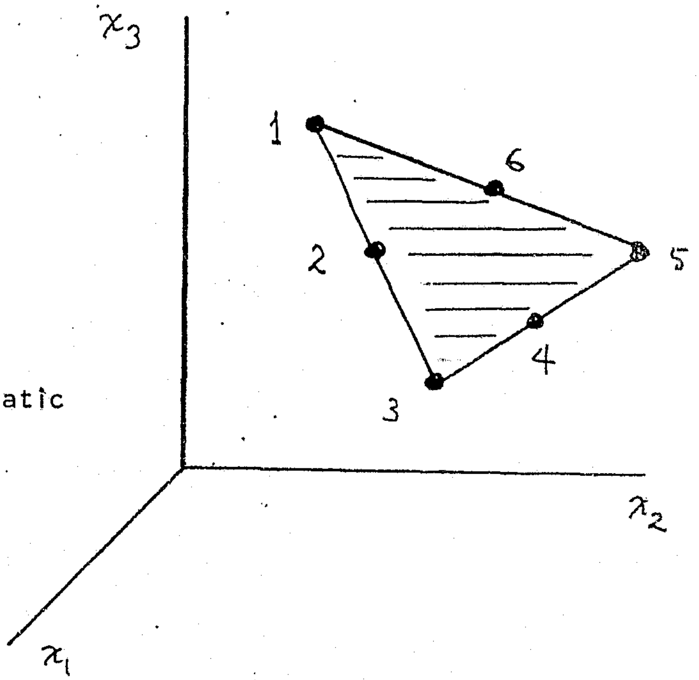
   For static problems: Incremental Loading (Euler's Method) plus Newton-Raphson corrections.

g. Required or preferred solution procedure:

   For dynamic problems: Incremental Loading. Divided central difference methods plus a Lax-Wendroff method.

ELEMENT ID: P-23

ELEMENT TYPE: Triangular membrane



ASSUMED DISPLACEMENT SHAPE: Quadratic

DESCRIPTION:
No. of nodes : 6
Degrees of freedom: 18
Coordinate system : Cartesian
Isotropic materials

REFERENCE: [114], [113], [107], [115], [117], [118].

VARIATION OF THIS ELEMENT: [63], [64], [65].

ADVANTAGES OR DISADVANTAGES: Advantages: Computational simplicity

Strain Displacement Equations:

$$\gamma_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{m,i}u_{m,j})$$

Discussion:

a. Stress-Strain equations: $\sigma^{ij} = \frac{1}{2}(\frac{\partial W}{\partial \gamma_{ij}} + \frac{\partial W}{\partial \gamma_{ji}} + pG^{ij}$

b. Terms retained in the strain energy: All

c. Derivations from other than strain energy considerations: Galerkin Methods, Conservation of Energy

d. Unique derivation techniques: See (c) above.

e. Lagrangian, Eulerian, convected: Lagrangian (convected on material)

f. Numerical integration schemes:

   For static problems: Incremental loading (Euler's Method) plus Newton-Raphson corrections.

g. Required or preferred solution procedure:

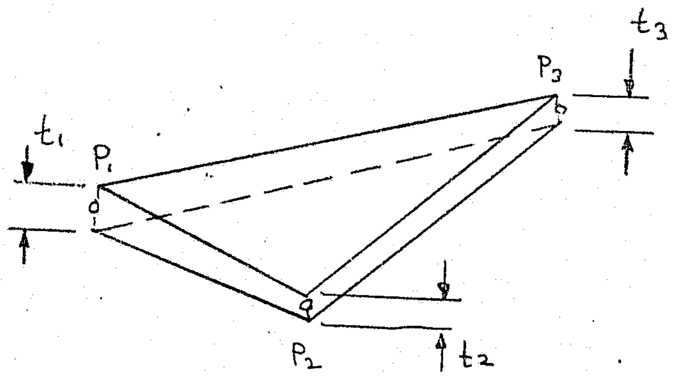
   For dynamic problems: Incremental Loading. Divided central difference methods plus a Lax-Wendroff method.

ELEMENT ID: P-24

ELEMENT TYPE: TRIM3
(triangular membrane element in 3 space)

ASSUMED DISPLACEMENT SHAPE: Linear

Figure:



DESCRIPTION: Number of nodes : 3
Degrees of freedom: u, v, w at each node, all together 9
Deformation type : stretching, shearing
Isotropic or anisotropic

REFERENCE: [94]

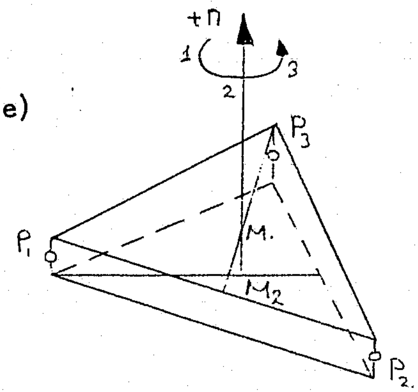
VARIATION OF THIS ELEMENT: [1], [2], [21], [60], [107], [113], [114],[115],[117], [118].
For application in large displacement and large strain the special element TRIMP3 was derived from TRIM3. This applies also to higher order elements of the TRIM family, which are not listed here (as TRIM6 and TRIMP6, TRIM10 and TRIMP10).

ADVANTAGES OR DISADVANTAGES:

DISCUSSION: Based on the concept of natural stress invariants for large rotations. Normally the application rate is low. But considering problems of large strains and large displacements this element is superior to even higher order elements, Ref. [95].

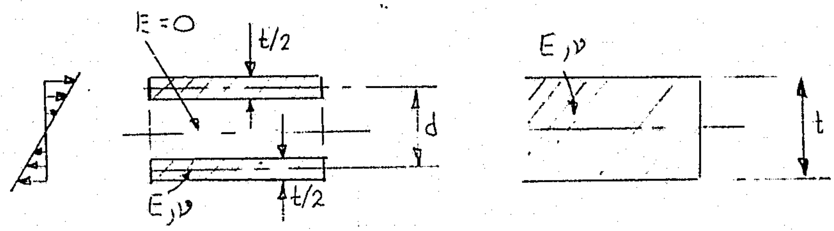
ELEMENT ID: P-25

ELEMENT TYPE: TRIB3
(Triangular bending element in 3-space)



ASSUMED DISPLACEMENT SHAPE: Linear in-plane
incomplete third order for bending

Figure:



DESCRIPTION: Number of nodes : 3
Degrees of freedom: u, v, w, ϕ_x, ϕ_y, ϕ_z, at each node
(ϕ. Angle of rotation), all together 18
Deformation type : stretching, shearing, bending without shear.
Isotropic or Anisotropic

REFERENCE: [97], [98]

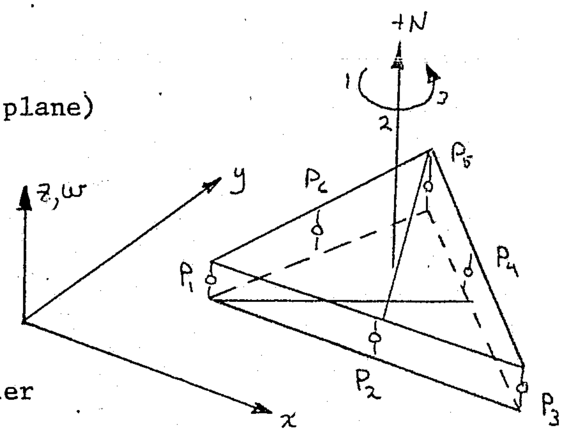
VARIATION OF THIS ELEMENT: An additional centrifugal stiffness part was developed for the application to rotating parts.

ADVANTAGES OR DISADVANTAGES: Advantages: Simple in application and use. No numerical integration, very fast element processing. Good experience in buckling and large displacements.

Disadvantages: Not fully compatible for non-plane structures. Curved structures have to be approximated by faceting thus involving a relatively large number of elements.

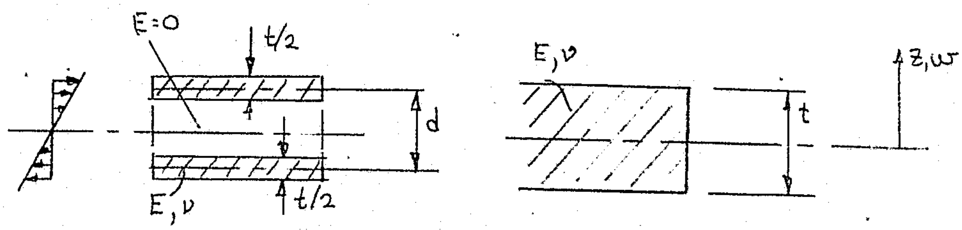
ELEMENT ID: P-26

ELEMENT TYPE: TUBA6 (TUBA3)
(Triangular bending element in plane)



ASSUMED DISPLACEMENT SHAPE: Complete fifth order

Figure:



DESCRIPTION:

Number of nodes : 6
Degrees of freedom : Vertices
w w, x w, y w, xx w, xy w, yy
3 4 5 6 7 8
Midside nodes:
w, n
9
n stands for a local direction normal to the edge; number beneath the symbols indicate the internal number of the corresponding degree of freedom; all together 21

Deformation type: bending
Isotropic or Anisotropic

REFERENCE: [99].

VARIATION OF THIS ELEMENT: Elimination of midside nodes leads to TUBA3.

ADVANTAGES OR DISADVANTAGES:

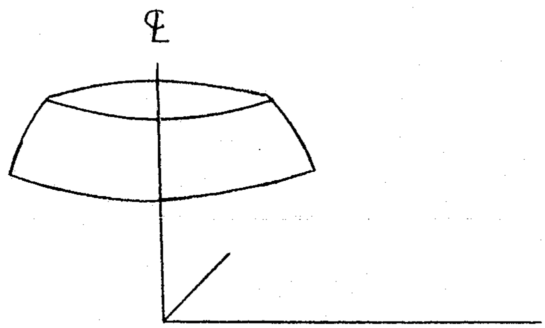
Advantages: More accurate then TRIB3 (based on the same effort), fully compatible.

Disadvantages: Jumps in thickness provide difficulties.

DISCUSSION: Based on the concept of natural stress invariants for large rotations.

ELEMENT ID: S-1

ELEMENT TYPE: Doubly Curved Shell of Revolution



(See next page also)

ASSUMED DISPLACEMENT SHAPE:

$$u_1 = a_1 + a_2 \xi$$

$$u_2 = a_3 + a_4 \xi + a_5 \xi^2 + a_6 \xi^3$$

Geometry:

$$\eta = \xi(1-\xi)(a_1 + a_2 \xi), \quad a_1 = \tan\beta_i', \quad a_2 = \tan\beta_i' + \frac{\eta_i''}{2}$$

DESCRIPTION:

This element is a doubly curved, isotropic shell of revolution restricted to axisymmetric loads. It has 2 ring nodes and 3 DOF (u_1, u_2, β) per node. Membrane and bending deformation states are included in the derivation. Geometric and material nonlinearities are considered.

REFERENCE: [42]

VARIATION OF THIS ELEMENT: [46], [47]

ADVANTAGES OR DISADVANTAGES:

ELEMENT ID: S-1

STRAIN DISPLACEMENT EQUATIONS:

$$\epsilon_{ss} = \epsilon_s + \zeta K_s, \quad \epsilon_{\theta\theta} = \epsilon_\theta + \zeta K_\theta$$

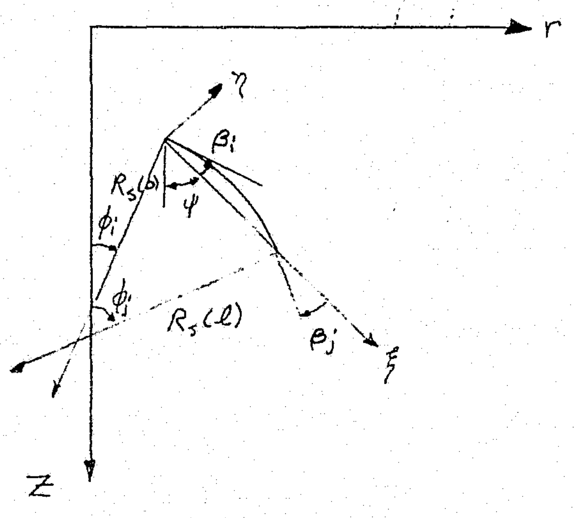
$$\epsilon_s = e_s + \frac{\chi^2}{2}; \quad \epsilon_\theta = e_\theta$$

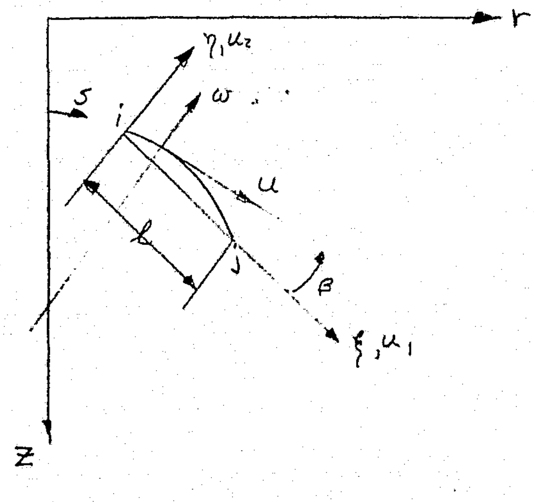
$$K_s = \chi_{,s} + \frac{e_s}{R_s} + \frac{\chi^2}{2R_s}, \quad K_\theta = \frac{\cos\phi}{r}\chi + \frac{e_\theta}{R_\theta} - \frac{\sin\phi}{2r}\chi^2$$

$$e_s = u_{,s} + \frac{w}{R_s}, \quad e_\theta = \frac{1}{r}(u\cos\phi + w\sin\phi), \quad \chi = \frac{u}{R_s} - w_{,s}$$

DISCUSSION:

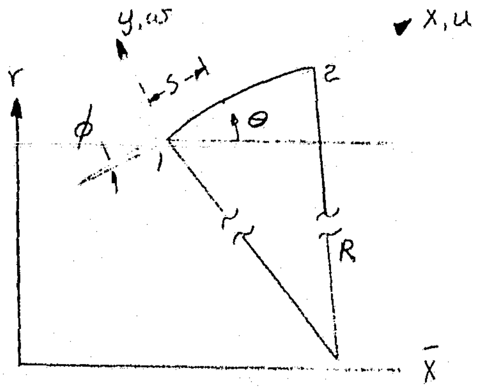
This element considers material nonlinearities which obey the von Mises yield criterion and isotropic hardening law. The incremental theory of plasticity is used. The linearly elastic and perfectly plastic conditions are included as special cases. The mechanical sublayer model is used for the elastic-plastic analyses. A Lagrangian derivation is employed and only quadratic products of displacements and/or displacement gradients are retained in the virtual work expression. The solution is obtained by an incremental stepwise procedure.





ELEMENT ID: S-1a

ELEMENT TYPE: Doubly Curved Shell of Revolution



REFERENCE: [46], [47]

VARIATIONS FROM BASIC ELEMENT: S-1

This element uses the same displacement function as element S-1, but the following form of the strain-displacement equations is employed

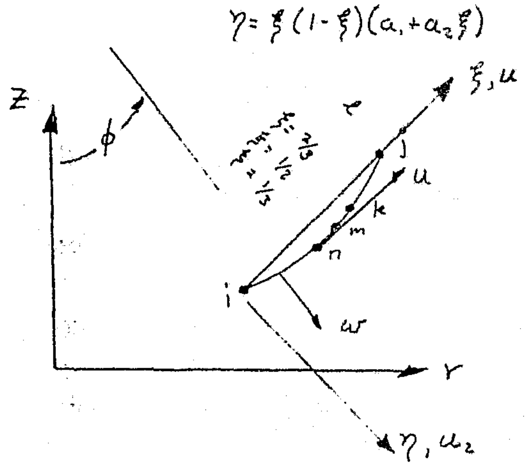
$$e_s = \frac{\partial u_c}{\partial s} + \frac{w_c}{R} + \frac{1}{2}(\chi_c)^2, \quad e_\psi = \frac{u_c}{r}\sin(\theta+\phi) + \frac{w_c}{r}\cos(\theta+\phi), \quad \chi_c = \frac{\partial w_c}{\partial s} - \frac{u_c}{R}$$

$$k_s = \frac{\partial \chi_c}{\partial s} = \frac{\partial^2 w_c}{\partial s^2} - \frac{1}{R}\frac{\partial u_c}{\partial s} - \frac{u_c \partial(\frac{1}{R})}{\partial s}, \quad k_\psi = \frac{\sin\theta}{r}\left(\frac{\partial w_c}{\partial s} - \frac{u_c}{R}\right)$$

Material nonlinearities are accounted for by the incremental theory of plasticity and the von Mises yield criterion. It is assumed to work harden according to an isotropic strain hardening criterion. The mechanical sublayer model is employed. Numerical integration is used to form the stiffness matrices. A direct solution using a sub-matrix form of the Crout-Banachiewicz reduction procedure was developed. A Lagrangian derivation is employed with all products of displacements and/or displacement gradients retained in the virtual work.

ELEMENT ID: S-2

ELEMENT TYPE: Doubly curved Shell of Revolution



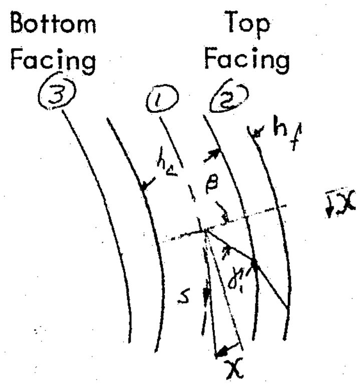
ASSUMED DISPLACEMENT SHAPE:

$$u_1 = \alpha_1 + \alpha_2 \xi + \alpha_3 \xi^2 + \alpha_4 \xi^3$$

$$u_2 = \alpha_5 + \alpha_6 \xi + \alpha_7 \xi^2 + \alpha_8 \xi^3$$

$$\gamma_c = \alpha_9 + \alpha_{10} \xi + \alpha_{11} \xi^2$$

$$\gamma_f = \alpha_{12} + \alpha_{13} \xi + \alpha_{14} \xi^2$$



DESCRIPTION:

This element is a doubly curved, isotropic, three layer sandwich shell of revolution. It has two external ring nodes with 5 DOF ($u_1, u_2, \gamma_c, \gamma_f, \chi$) per node and three internal ring nodes with DOF (u_1), (γ_c, γ_f), and (u_1) respectively. Membrane, bending, and shear stiffnesses are included in the derivation. Geometric and material nonlinearities are considered.

REFERENCE: [49]

VARIATION OF THIS ELEMENT: S-2a

ADVANTAGES OR DISADVANTAGES:

It may be difficult to incorporate this element in a general purpose program because of the non-typical nodal degrees of freedom. This element satisfies completeness and compatibility requirements.

ELEMENT ID: S-2

STRAIN DISPLACEMENT EQUATIONS:

$$\begin{Bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ 2\epsilon_{13}\end{Bmatrix}_K = \begin{Bmatrix}\epsilon_{11}^{o}\\ \epsilon_{22}^{o}\\ 2\epsilon_{13}\end{Bmatrix}_K + \zeta_K \begin{Bmatrix}K_{11}^{1}\\ K_{22}^{1}\\ 0\end{Bmatrix}_K + \begin{Bmatrix}\eta_{11}^{o}\\ \eta_{22}^{o}\\ 2\eta_{13}\end{Bmatrix}_K + \zeta_K \begin{Bmatrix}K_{11}^{2}\\ K_{22}^{2}\\ 0\end{Bmatrix}_K \qquad K = 1, 2, 3$$

$$\begin{Bmatrix}e_{11}^{o}\\ e_{22}^{o}\\ 2e_{13}\end{Bmatrix}_1 = \begin{Bmatrix}e_{1c}\\ e_{2c}\\ \gamma_c\end{Bmatrix}, \quad \begin{Bmatrix}e_{11}^{o}\\ e_{22}^{o}\\ 2e_{13}\end{Bmatrix}_{\substack{2\\3}} = \begin{Bmatrix}e_{1c} \pm \frac{d}{2}\left[K_{1x} + r_c K_{1c} + r_f K_{1f}\right]\\ e_{2c} \pm \frac{d}{2}\left[K_{2x} + r_c K_{2c} + r_f K_{2f}\right]\\ \gamma_f\end{Bmatrix}$$

$$\begin{Bmatrix}K_{11}^{1}\\ K_{22}^{1}\end{Bmatrix}_1 = \begin{Bmatrix}K_{1x} + K_{1c}\\ K_{2x} + K_{2c}\end{Bmatrix}, \quad \begin{Bmatrix}K_{11}^{1}\\ K_{22}^{1}\end{Bmatrix}_{\substack{2\\3}} = \begin{Bmatrix}K_{1x} + K_{1f}\\ K_{2x} + K_{2f}\end{Bmatrix}, \quad \begin{Bmatrix}\eta_{11}^{o}\\ \eta_{22}^{o}\\ 2\eta_{13}^{o}\end{Bmatrix}_1 = \begin{Bmatrix}\frac{1}{2}(e_{1c}^2 + \chi^2)\\ \frac{1}{2}(e_{2c})^2\\ (\chi + \gamma_c)e_{1c}\end{Bmatrix}$$

$$\begin{Bmatrix}\eta_{11}^{o}\\ \eta_{22}^{o}\\ 2\eta_{13}^{o}\end{Bmatrix}_{\substack{2\\3}} = \begin{Bmatrix}\frac{1}{2}(e_{1c}^2 + \chi^2) \pm \frac{d}{2}(K_{1x} + r_c K_{1c} + r_f K_{1f})e_{1c} \pm \frac{d}{4R_1}(\chi^2 - r_c\gamma_c^2 + r_f\gamma_f^2)\\ \frac{1}{2}e_{2c}^2 \pm \frac{d}{2}(K_{2x} + r_c K_{2c} + r_f K_{2f})e_{2c} \pm \frac{d}{4r}\left[r_c(\chi + \gamma_c)^2 + r_f(\chi + \gamma_f)^2\right]\sin\phi\\ (\chi + \gamma_f)e_{1c}\end{Bmatrix}$$

$$\begin{Bmatrix}K_{11}^{2}\\ K_{22}^{2}\end{Bmatrix}_1 = \begin{Bmatrix}(K_{1x} + K_{1c})e_{1c} + \frac{1}{2R_1}(\chi^2 - \gamma_c^2)\\ (K_{2x} + K_{2c})e_{2c} - \frac{1}{2r}(\chi + \gamma_c)^2\sin\phi\end{Bmatrix}, \quad \begin{Bmatrix}K_{11}^{2}\\ K_{22}^{2}\end{Bmatrix}_{\substack{2\\3}} = \begin{Bmatrix}(K_{1x} + K_{1f})e_{1c} + \frac{1}{2R_1}(\chi^2 - \gamma_f^2)\\ (K_{2x} + K_{2f})e_{2c} - \frac{1}{2r}(\chi - \gamma_f)^2\sin\phi\end{Bmatrix}$$

where: $e_{1c} = \frac{\partial u_1^o}{\partial s} + \frac{\omega}{R_1}$; $e_{2c} = \frac{1}{r}(u_1^o\cos\phi + \omega\sin\phi)$, $K_{1x} = \frac{\partial \chi}{\partial s}$, $K_{2x} = \frac{\chi}{r}\cos\phi$

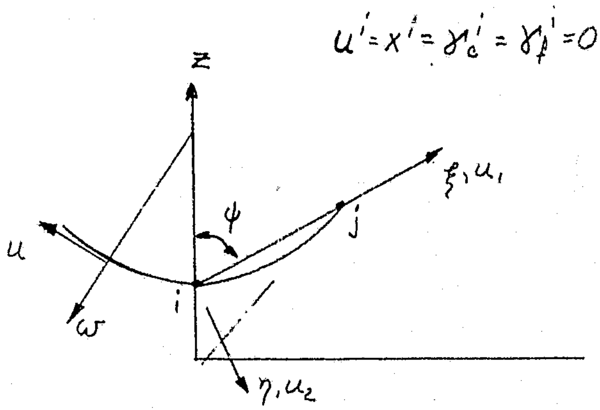
$\chi = \frac{u_1^o}{R_1} - \frac{\partial \omega}{\partial s}$, $K_{1c} = \frac{\partial \gamma_c}{\partial s}$; $K_{2c} = \frac{\gamma_c}{r}\cos\phi$, $K_{1f} = \frac{\partial \gamma_f}{\partial s}$; $K_{2f} = \frac{\gamma_f}{r}\cos\phi$

$r_c = \frac{h_c}{d}$, $r_f = \frac{h_f}{d}$, $r = R_2\sin\phi$, $ds = R_1 d\phi$

This element considers material nonlinearities which obey the von Mises yield criterion and isotropic hardening law. The incremental theory of plasticity is used. The core material is elastic whereas the facings may be inelastic. An incremental variational form based on a moving reference configuration is used to derive element equilibrium equations. The incremental Lagrangian forms of strain displacement equations include terms corresponding to large rotations. Only quadratic products of displacements and/or displacement gradients are retained in the virtual work. The solution is obtained by an incremental stepwise procedure with post-buckling solutions obtained by augmenting the structure to obtain a positive definite stiffness matrix.

ELEMENT ID: S-2a

ELEMENT TYPE: Doubly curved shell of revolution cap element



REFERENCE: [49]

VARIATIONS FROM BASIC ELEMENT: S-2

This is the frustum element of S-2 specialized to a cap element. The assumed displacement shapes have the form

$$u_1 = -\alpha_3 \cos\psi + \alpha_4 \xi + \alpha_5 \xi^2 + \alpha_6 \xi^3$$

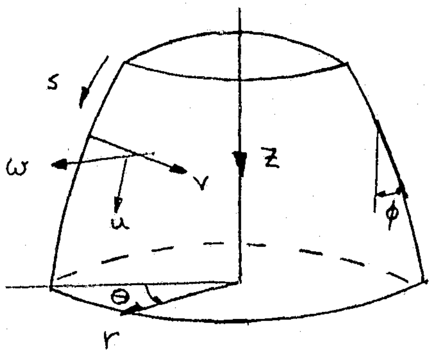
$$u_2 = \alpha_3 \sin\psi + \alpha_4 \tan\alpha_i \xi + \alpha_7 \xi^2 + \alpha_8 \xi^3$$

$$\gamma_c = \alpha_{10} \xi + \alpha_{11} \xi^2 ,$$

$$\gamma_f = \alpha_{13} \xi + \alpha_{14} \xi^2$$

ELEMENT ID: S-3

ELEMENT TYPE: Doubly Curved Shell of Revolution



$\phi = a_1 + a_2 s + a_3 s^2$

ASSUMED DISPLACEMENT SHAPE:

$$u = \sum_{i=0}^{IA} (\alpha_5^i + \alpha_6^i s) \cos i\theta + \sum_{i=0}^{IB} (\bar{\alpha}_5^i + \bar{\alpha}_6^i s) \sin i\theta$$

$$v = \sum_{i=0}^{IA} (\alpha_7^i + \alpha_8^i s) \sin i\theta + \sum_{i=0}^{IB} (\bar{\alpha}_7^i + \bar{\alpha}_8^i s) \cos i\theta$$

$$w = \sum_{i=0}^{IA} (\alpha_1^i + \alpha_2^i s + \alpha_3^i s^2 + \alpha_4^i s^3) \cos i\theta + \sum_{i=0}^{IB} (\bar{\alpha}_1^i + \bar{\alpha}_2^i s + \bar{\alpha}_3^i s^2 + \bar{\alpha}_4^i s^3) \sin i\theta$$

DESCRIPTION:

This element is a doubly curved orthotropic shell of revolution developed for arbitrary nonaxisymmetic loads. It has 2 ring nodes and 4 DOF (u, v, w, rotation) at each node. Membrane and bending deformation states are included in the derivation. Only geometric nonlinearities are considered.

REFERENCE: [50]

VARIATION OF THIS ELEMENT: S-3a, S-3b, S-3c

ADVANTAGES OR DISADVANTAGES:

In a later paper [53] the authors show that fourth order products of displacement and/or displacement gradients should be retained in the strain energy to improve solutions.

ELEMENT ID: S-3

STRAIN DISPLACEMENT EQUATIONS:

$$\epsilon_s = \hat{e}_s + \frac{1}{2}\hat{e}_{13}^2, \quad \epsilon_\theta = \hat{e}_\theta + \frac{1}{2}\hat{e}_{23}^2, \quad \epsilon_{s\theta} = \hat{e}_{s\theta} + \hat{e}_{13}\hat{e}_{23}$$

$$\chi_s = -\frac{\partial \hat{e}_{13}}{\partial s}, \quad \chi_\theta = -\frac{1}{r}\frac{\partial \hat{e}_{23}}{\partial \theta} - \frac{1}{r}\sin\phi\,\hat{e}_{13}, \quad \chi_{s\theta} = -\frac{1}{r}\frac{\partial \hat{e}_{13}}{\partial \theta} + \frac{1}{r}\sin\phi\,\hat{e}_{23} - \frac{\partial \hat{e}_{23}}{\partial s}$$

$$\hat{e}_s = \frac{\partial u}{\partial s} - \phi' w, \quad \hat{e}_\theta = \frac{1}{r}\left[\frac{\partial v}{\partial \theta} + u\sin\phi + w\cos\phi\right]$$

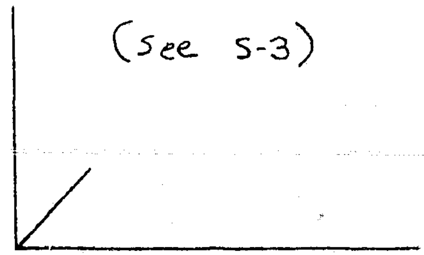
$$\hat{e}_{s\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r}\sin\phi + \frac{\partial v}{\partial s}, \quad \hat{e}_{13} = \frac{\partial w}{\partial s} + u\phi', \quad \hat{e}_{23} = \frac{1}{r}\frac{\partial w}{\partial \theta} - \frac{v}{r}\cos\phi$$

DISCUSSION:

This element employs a linear stress-strain relation in the element derivation. Cubic products of displacements and/or displacement gradients are retained in the strain energy, but are treated as an additional force column. The element is developed through the use of the non-shallow shell theory of Novozilov. The displacement function is that of [51] as extended in [52]. The stiffness matrix is evaluated through numerical integration over the length and the arbitrary loading represented by a Fourier series in the circumferential direction. A Lagrangian derivation is employed with the resulting equilibrium equations solved by one of three methods: load increment, iteration, or a combination of the two.

ELEMENT ID: S-3a

ELEMENT TYPE: Doubly curved Shell of Revolution



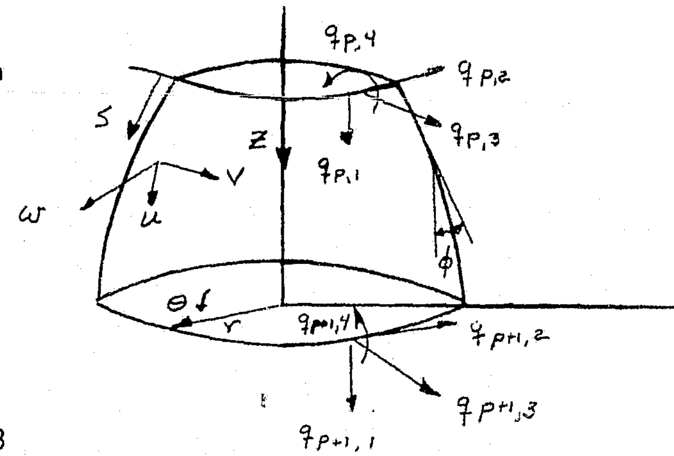
REFERENCE: [53]

VARIATIONS FROM BASIC ELEMENT: S-3

This element is identical to that of S-3, except that the fourth order products of displacements and/or displacement gradients are retained in the strain energy. Several examples are included that illustrate the improved solutions.

ELEMENT ID: S-3b

ELEMENT TYPE: Doubly Curved Shell of Revolution



REFERENCE: [61]

VARIATIONS FROM BASIC ELEMENT: S-3

This element apparently uses the same cubic displacement function as S-3 but the generalized displacements are given explicitly as

$$w = \sum_{i=0} \left[\left(1 - 3\frac{s^2}{L^2} + 2\frac{s^3}{L^3}\right)\bar{q}_3^{\,i} + \left(s - 2\frac{s^2}{L} + \frac{s^3}{L^2}\right)\left(q_4^{\,i} - q_1^{\,i}\phi'\right) + \left(3\frac{s^2}{L^2} - 2\frac{s^3}{L^3}\right)\bar{q}_7^{\,i} + \left(-\frac{s^2}{L} + \frac{s^3}{L^2}\right)\left(q_8^{\,i} - \bar{q}_5^{\,i}\phi_1'\right) \right] \cos i\theta$$

$$u = \sum_{i=0} \left[\left(1 - \frac{s}{L}\right)\bar{q}_1^{\,i} + \bar{q}_5^{\,i}\frac{s}{L} \right] \cos i\theta$$

$$v = \sum_{i=0} \left[\left(1 - \frac{s}{L}\right)\bar{q}_2^{\,i} + \bar{q}_6^{\,i}\frac{s}{L} \right] \cos i\theta$$

The w displacement is assumed to be

$$w = \sum_{i=0} \left[\left(1 - \frac{s}{L}\right)\bar{q}_3^{\,i} + \bar{q}_7^{\,i}\frac{s}{L} \right] \cos i\theta$$

in the strain energy contributions introduced by nonlinear theory, and

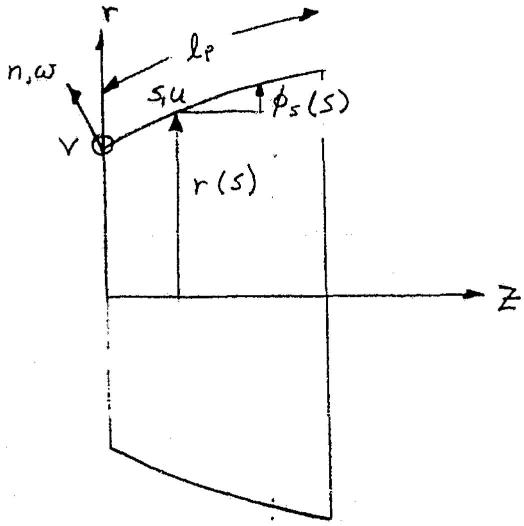
$$\bar{q}_1 = q_1 \cos\phi_o + q_3 \sin\phi_o, \qquad \bar{q}_3 = -q_1 \sin\phi_o + q_3 \cos\phi_o$$

$$\bar{q}_5 = q_5 \cos\phi_1 + q_7 \sin\phi_1, \qquad \bar{q}_7 = -q_5 \sin\phi_1 + q_7 \cos\phi_1$$

The element is a doubly curved orthotropic shell of revolution developed for arbitrary nonaxisymmetric dynamic loads. In other respects, it is similar to element S-3 except that fourth order products of displacements and/or displacement gradients are retained in the strain energy as recommended in element S-3a.

ELEMENT ID: S-3c

ELEMENT TYPE: Doubly Curved Shell of Revolution



REFERENCE: [60]

VARIATIONS FROM BASIC ELEMENT: S-3

This element uses the cosine harmonic displacement function as shown in S-3 (i.e., $\sum_{i=0}^{IA}$ terms), but obtains an eiganvalue solution for the buckling load. It is a doubly curved isotropic shell of revolution developed for nonaxisymmetric eiganvectors. It has 2 ring nodes and 4 DOF (u, v, w, rotation) at each node. Membrane and bending deformation states are included in the derivation. Only geometric nonlinearities are considered. The Sanders' strain displacement equations for moderate bending are given as

$$\epsilon_{ss} = (u_{,s} - \omega\phi_{,s}) + \tfrac{1}{2}(\Psi_s^2 + \Psi^{*2})$$

$$\epsilon_{\theta\theta} = \left(\tfrac{1}{r}\right)(u\sin\phi + v_{,\theta} + \omega\cos\phi) + \tfrac{1}{2}(\Psi_\theta^2 + \Psi^{*2})$$

$$\epsilon_{s\theta} = \left(\tfrac{1}{2r}\right)(r v_{,s} + u_{,\theta} - v\sin\phi) + \tfrac{1}{2}\Psi_s\Psi_\theta$$

$$K_{ss} = \Psi_{s,s} \qquad K_{\theta\theta} = \tfrac{1}{r}(\Psi_{\theta,\theta} + \Psi_s\sin\phi)$$

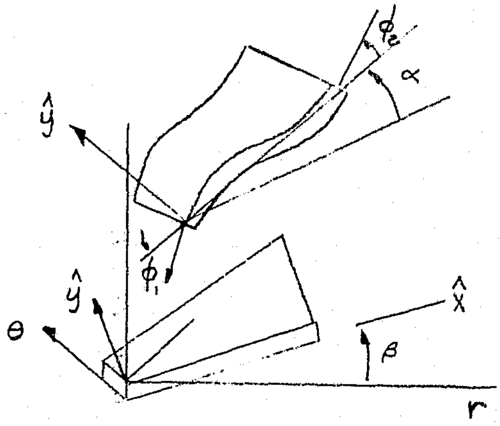
$$K = \tfrac{1}{2}\left[\Psi_{\theta,s} + \tfrac{1}{r}\Psi_{s,\theta} - \Psi_\theta\frac{\sin\phi}{r} + \left(\phi_{,s} + \frac{\cos\phi}{r}\right)\Psi^*\right]$$

where $\Psi_s = -(\omega_{,s} + u\phi_{,s})$, $\Psi_\theta = -\left(\tfrac{1}{r}\right)(\omega_{,\theta} - v\cos\phi)$, $\Psi^* = \tfrac{1}{2r}(r v_{,s} + v\sin\phi - u_{,\theta})$

By dropping the terms with asterisks, the Donnell-Mushtari-Vlasov strain-displacement relations are obtained. A linear stress-strain relation is used. A Lagrangian derivation is employed with quadratic products of displacements and/or displacement gradients retained in the strain energy.

ELEMENT ID: S-4

ELEMENT TYPE: Conical Shell of Revolution



ASSUMED DISPLACEMENT SHAPE:

For convected coordinates, the deformation displacements are the nodal rotations relative to the $\hat{x}$ axis

$$\hat{\phi}(\hat{x}) = \frac{\hat{\phi}_1}{\ell_2}\left(\ell^2 - 4\ell\hat{x} - 3\hat{x}^2\right) + \frac{\hat{\phi}_2}{\ell^2}\left(3\hat{x}^2 - 2\ell\hat{x}\right)$$

$$\hat{u}_x = a_0 + a_1\hat{x}$$

DESCRIPTION:

This element is an isotropic conical shell element developed for a large displacement small strain, elastic-plastic dynamic problem. It has 2 ring nodes and 3 DOF (u_x, u_y, ϕ) at each node. Convected coordinates are used to derive the elemental matrices. Membrane and bending deformation states are included in the derivation. Both geometric and material non-linearities are included.

REFERENCE: [54]

VARIATION OF THIS ELEMENT:

ADVANTAGES OR DISADVANTAGES:

The method is stated as being computationally efficient.

ELEMENT ID: S-4

STRAIN DISPLACEMENT EQUATIONS:

$$\hat{\varepsilon}_x = \frac{d\hat{u}_x^{def}}{d\hat{x}} - \hat{y}\,\frac{d\hat{\phi}(\hat{x})}{d\hat{x}}$$

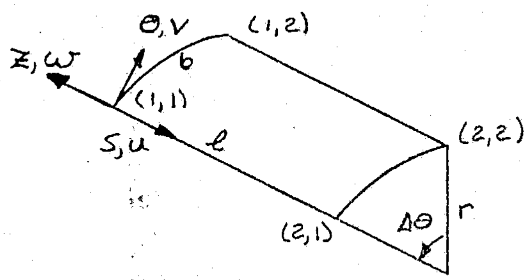
$$\varepsilon_\theta = \frac{1}{r}\left(u_r - \hat{y}\cos\beta\,\frac{\partial u_y}{\partial x}\right)$$

DISCUSSION:

This somewhat brief paper indicates the element can account for material nonlinearities by using numerical quadrature through the thickness. Because convected coordinates are used, the geometric nonlinearities that arise are accounted for by transformation between the convected and global systems. The strains are linearly related to deformation displacements relative to the convected coordinate system. The dynamic equations of equilibrium are solved by a central difference explicit integration procedure which uses total nodal forces.

ELEMENT ID: S-5

ELEMENT TYPE: Cylindrical Shell Element of Rectangular Plan-form



ASSUMED DISPLACEMENT SHAPE:

$$w(s, r\theta) = \sum_{i=1}^{2}\sum_{j=1}^{2}\Big[H_{0i}^{(1)}(s)\,H_{0j}^{(1)}(r\theta)\,w_{ij} + H_{1i}^{(1)}(s)\,H_{0j}^{(1)}(r\theta)\,w_{x_{ij}} + H_{0i}^{(1)}(s)\,H_{1j}^{(1)}(r\theta)\,w_{y_{ij}} + H_{1i}^{(1)}(s)\,H_{1j}^{(1)}(r\theta)\,w_{xy_{ij}}\Big]$$

$$H_{01}^{(1)}(s) = (2s^3 - 3\ell s^2 + \ell^3)/\ell^3 \qquad H_{11}^{(1)}(s) = (s^3 - 2\ell s^2 + \ell^2 s)/\ell^2$$

$$H_{02}^{(1)}(s) = -(2s^3 - 3\ell s^2)/\ell^3 \qquad H_{12}^{(1)}(s) = (s^3 - \ell s^2)/\ell^2$$

Similarly for the θ direction, replace s by $r\theta$, and l by b. Similar expressions for u and v.

DESCRIPTION:

This element is an isotropic cylindrical shell element of rectangular planform with 4 nodes and 12 DOF $(u, v, w, u_x, v_x, w_x, u_y, v_y, w_y, u_{xy}, v_{xy}, w_{xy})$ per node. Membrane and bending deformation states are accounted for. Displacements are approximated by products of one-dimensional cubic Hermite interpolation polynomials. Only geometric nonlinearities are considered.

REFERENCE: [16]

VARIATION OF THIS ELEMENT: See also P-2, P-2a, P-12, S-5a

ADVANTAGES OR DISADVANTAGES:

The bicubic interpolation functions for u, v, and w admit to six linearly independent displacement states of very little strain energy. Consequently, this element resolves the rigid body mode problem. The selected nodal DOF may not be compatible with other elements in a general purpose structural analysis computer program.

ELEMENT ID: S-5

STRAIN DISPLACEMENT EQUATIONS:

$$\epsilon_s = \frac{\partial u}{\partial s} + \frac{1}{2}\left(\frac{\partial w}{\partial s}\right)^2 - z\left(\frac{\partial^2 w}{\partial s^2}\right)$$

$$\epsilon_\theta = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{w}{r} + \frac{1}{2r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 - \frac{z}{r^2}\left(\frac{\partial^2 w}{\partial \theta^2} - \frac{\partial r}{\partial \theta}\right)$$

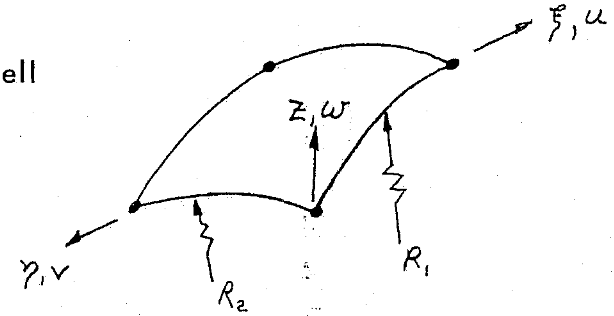
$$\gamma_{s\theta} = \frac{\partial v}{\partial s} + \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial w}{\partial s}\frac{\partial w}{\partial \theta} - \frac{2z}{r}\left(\frac{\partial^2 w}{\partial s\,\partial \theta} - \frac{\partial v}{\partial s}\right)$$

DISCUSSION:

This element employs a linear stress-strain relation with all combinations of displacements and/or displacement gradients retained in the strain energy. A Lagrangian derivation is employed with a numerical solution obtained by direct minimization of the total potential energy.

ELEMENT ID: S-5a

ELEMENT TYPE: Doubly curved
Rectangular Shell



REFERENCE: [55]

VARIATIONS FROM BASIC ELEMENT: S-5

This element employs the displacement function of Ref. [16] for a doubly curved shell element. Only geometric nonlinearities are considered. The shell strain displacement equations are

$$\varepsilon_1^{z} = u_{\xi} + r_1 \omega + \frac{1}{2}(\omega_{\xi})^2 - r_1(u)(\omega_{\xi}) - z(r_1 u_{\xi} - \omega_{\xi\xi})$$

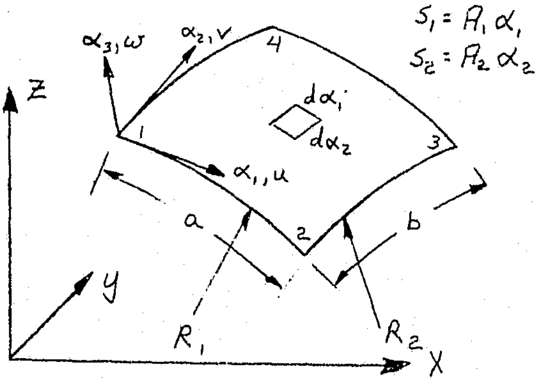
$$\varepsilon_2^{z} = v_{\eta} + r_2 \omega + \frac{1}{2}(\omega_{\eta})^2 - r_2(v)(\omega_{\eta}) - z(r_2 v_{\eta} - \omega_{\eta\eta})$$

$$\varepsilon_{12}^{z} = v_{\xi} + u_{\eta} + (\omega_{\xi})(\omega_{\eta}) - r_1(u)(\omega_{\eta}) - r_2(v)(\omega_{\xi}) - z(r_2 v_{\xi} + r_1 u_{\eta} - 2\omega_{\xi\eta})$$

where $r_1 = 1/R_1$ and $r_2 = 1/R_2$ are constant. The curvilinear coordinates are ξ, η with subscripts denoting derivatives. A systematic integration method is developed that simplifies the element formulation. It can account for a non-uniform distribution of initial stress. Apparently cubic products of displacements and/or displacement gradients are retained in the strain energy, but since an eiganvalue solution is performed to determine dynamic modes and frequencies a stress state is assumed. As used in the context of this document, this is equivalent to an initial stress state with quadratic product of displacements and/or displacement gradients retained in the strain energy. A Lagrangian derivation is employed with an eigansolution performed to find dynamic modes and frequencies. The linear stiffness matrix is derived in Ref. [56] based on the strain displacement equations of [57].

ELEMENT ID: S-6

ELEMENT TYPE: Doubly Curved Rectangular Shell



ASSUMED DISPLACEMENT SHAPE:

$$u = \left[(\alpha_1 - a)(\alpha_2 - b)u_1 - \alpha_1(\alpha_2 - b)u_2 + \alpha_1\alpha_2 u_3 - (\alpha_1 - a)\alpha_2 u_4\right]/ab$$

$$v = \left[(\alpha_1 - a)(\alpha_2 - b)v_1 - \alpha_1(\alpha_2 - b)v_2 + \alpha_1\alpha_2 v_3 - (\alpha_1 - a)\alpha_2 v_4\right]/ab$$

w = the product of one dimensional cubic Hermite interpolation polynomials similar to that of S-5. It is given explicitly on the following pages.

DESCRIPTION:

This element is an isotropic doubly curved shell element of rectangular shape with 4 nodes and 6 DOF ($u, v, w, \theta_{\alpha_1}, \theta_{\alpha_2}, \theta_{\alpha_1\alpha_2}$) per node. Membrane and bending deformation states are accounted for. Displacements are approximated by Lagrangian and Hermitian interpolation polynomials. Only geometric nonlinearities are considered.

REFERENCE: [58]

VARIATION OF THIS ELEMENT: P-6

ADVANTAGES OR DISADVANTAGES:

This element violates both the postulated requirements that rigid body motion be included in the assumed displacement shape and that displacements be continuous across interelement boundaries. The authors found these factors had little influence on the solutions obtained.

ELEMENT ID: S-6

STRAIN DISPLACEMENT EQUATIONS: Extension of Novozhilov equations [54]

Deep shell eqs. given on following pages.

DISCUSSION:

This element employs a linear stress-strain relation with only quadratic products of displacements and/or displacement gradients retained in the strain energy. The element is developed through the use of an extension of the shell equations of Novoshilov [59]. Explicit forms of the stiffness equations are obtained in related references. A Lagrangian derivation is employed with an eigansolution of the resulting equations. A Guyan type reduction is applied to the initial stress stiffness matrix on the structure level and the results compared to unreduced solutions.

$$
\begin{aligned}
w = \Big[& (a^3 - 2\alpha_1^3 - 3a\alpha_1^2)(b^3 + 2\alpha_2^3 - 3b\alpha_2^2)\, w_1 + (3a\alpha_1^2 - 2\alpha_1^3)(b^3 + 2\alpha_2^3 - 3b\alpha_2^2)\, w_2 \\
& + (3a\alpha_1^2 - 2\alpha_1^3)(3b\alpha_2^2 - 2\alpha_2^3)\, w_3 + (a^3 + 2\alpha_1^3 - 3a\alpha_1^2)(3b\alpha_2^2 - 2a_2^3)\, w_4 \\
& + a\alpha_1(\alpha_1 - a)^2 (b^3 + 2\alpha_2^3 - 3b\alpha_2^2)\, \theta\alpha_{1_1} + a(\alpha_1^3 - a\alpha_1^2)(b^3 + 2\alpha_2^3 - 3b\alpha_2^2)\, \theta\alpha_{1_2} \\
& + a(\alpha_1^3 - a\alpha_1^2)(3b\alpha_2^2 - 2\alpha_2^3)\, \theta\alpha_{1_3} + a(\alpha_1 - a)^2 \alpha_1 (3b\alpha_2^2 - 2\alpha_2^3)\, \theta\alpha_{1_4} \\
& + b(a^3 + 2\alpha_1^3 - 3a\alpha_1^2)\, \alpha_2(\alpha_2 - b)^2\, \theta\alpha_{2_1} + b(3a\alpha_1^2 - 2\alpha_1^3)\, \alpha_2(\alpha_2 - b)^2\, \theta\alpha_{2_2} \\
& + b(3a\alpha_1^2 - 2\alpha_1^3)(\alpha_2^3 - b\alpha_2^2)\, \theta\alpha_{2_3} + b(a^3 + 2\alpha_1^3 - 3a\alpha_1^2)(\alpha_2^3 - b\alpha_2^2)\, \theta\alpha_{2_4} \\
& + ab\alpha_1\alpha_2(\alpha_1 - a)^2(\alpha_2 - b)^2\, \theta\alpha\alpha_1 + ab\alpha_1\alpha_2(\alpha_1^2 - a\alpha_1)(\alpha_2 - b)^2\, \theta\alpha\alpha_2 \\
& + ab\alpha_1\alpha_2(\alpha_1^2 - a\alpha_1)(\alpha_2^2 - b\alpha_2)\, \theta\alpha\alpha_3 + ab\alpha_1\alpha_2(\alpha_1 - a)^2(\alpha_2^2 - b\alpha_2)\, \theta\alpha\alpha_4 \Big] \\
& \div a^3 b^3
\end{aligned}
$$

$$
\varepsilon_{\alpha_1} = \frac{\partial u}{\partial s_1} + \frac{w}{R_1} + \frac{1}{2}\left(\frac{\partial v}{\partial s_1}\right)^2 + \frac{1}{2}\left(\frac{u}{R_1} - \frac{\partial w}{\partial s_1}\right)^2
$$

$$
\varepsilon_{\alpha_2} = \frac{\partial v}{\partial s_2} + \frac{w}{R_2} + \frac{1}{2}\left(\frac{\partial u}{\partial s_2}\right)^2 + \frac{1}{2}\left(\frac{v}{R_2} - \frac{\partial w}{\partial s_2}\right)^2
$$

$$
\begin{aligned}
\omega = & \frac{\partial u}{\partial s_2} + \frac{\partial v}{\partial s_1} - \left(\frac{\partial u}{\partial s_1}\right)\left(\frac{\partial v}{\partial s_1}\right) - \left(\frac{\partial v}{\partial s_1}\right)\left(\frac{w}{R_1}\right) + \left(\frac{\partial w}{\partial s_1}\right)\left(\frac{\partial w}{\partial s_2}\right) - \left(\frac{v}{R_2}\right)\left(\frac{\partial w}{\partial s_1}\right) \\
& - \left(\frac{u}{R_1}\right)\left(\frac{\partial w}{\partial s_2}\right) + \left(\frac{u}{R_1}\right)\left(\frac{v}{R_2}\right) - \left(\frac{\partial u}{\partial s_2}\right)\left(\frac{\partial v}{\partial s_2}\right) - \left(\frac{\partial u}{\partial s_2}\right)\left(\frac{w}{R_2}\right)
\end{aligned}
$$

$$
\varepsilon_{\alpha_1}^{(z)} = \varepsilon_{\alpha_1} + K_1 z + \lambda_1 z^2
$$

$$
\varepsilon_{\alpha_2}^{(z)} = \varepsilon_{\alpha_2} + K_2 z + \lambda_2 z^2
$$

$$
\omega^{(z)} = \omega + K_{12} z + \lambda_{12} z^2
$$

$$K_1 = \frac{\partial^2 w}{\partial s_1^2}\left(-1+\frac{\partial u}{\partial s_1}+\frac{w}{R_1}\right)-\frac{1}{R_1}\left[\frac{w}{R_1}+\frac{1}{2}\left\{\left(\frac{\partial u}{\partial s_2}+\frac{\partial v}{\partial s_1}\right)^2+\left(\frac{u}{R_1}-\frac{\partial w}{\partial s_1}\right)^2-\left(\frac{v}{R_2}-\frac{\partial w}{\partial s_2}\right)^2\right\}\right.$$

$$\left.-\left(\frac{\partial w}{\partial s_1}\right)^2+\left(\frac{\partial u}{\partial s_1}\right)^2+\left(\frac{\partial v}{\partial s_1}\right)^2+\left(\frac{\partial u}{\partial s_1}\right)\left(\frac{w}{R_1}\right)+\left(\frac{\partial w}{\partial s_1}\right)\left(\frac{u}{R_1}\right)\right]$$

$$K_2 = \frac{\partial^2 w}{\partial s_1^2}\left(-1+\frac{\partial v}{\partial s_2}+\frac{w}{R_2}\right)-\frac{1}{R_2}\left[\frac{w}{R_2}-\frac{1}{2}\left\{\left(\frac{\partial u}{\partial s_2}+\frac{\partial v}{\partial s_1}\right)^2+\left(\frac{u}{R_1}-\frac{\partial w}{\partial s_1}\right)^2\right.\right.$$

$$\left.\left.+\left(\frac{v}{R_2}-\frac{\partial w}{\partial s_2}\right)^2\right\}-\left(\frac{\partial w}{\partial s_2}\right)^2+\left(\frac{\partial v}{\partial s_2}\right)^2+\left(\frac{\partial u}{\partial s_2}\right)^2+\left(\frac{\partial v}{\partial s_2}\right)\left(\frac{w}{R_2}\right)+\left(\frac{\partial w}{\partial s_2}\right)\left(\frac{v}{R_2}\right)\right]$$

$$\lambda_1 = \frac{1}{2}\left\{\frac{\partial^2 w}{\partial s_1 \partial s_2}+\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\left(\frac{\partial v}{\partial s_1}\right)\right\}^2$$

$$\lambda_2 = \frac{1}{2}\left\{\frac{\partial^2 w}{\partial s_1 \partial s_2}+\left(\frac{1}{R_2}-\frac{1}{R_1}\right)\left(\frac{\partial u}{\partial s_2}\right)\right\}^2$$

$$K_{12} = 2\left\{\frac{\partial^2 w}{\partial s_1 \partial s_2}\left(-1+\frac{\partial u}{\partial s_1}+\frac{\partial v}{\partial s_2}+\frac{w}{R_1}+\frac{w}{R_2}\right)+\frac{\partial^2 u}{\partial s_1 \partial s_2}\left(\frac{\partial w}{\partial s_1}-\frac{u}{R_1}\right)+\frac{\partial^2 v}{\partial s_1 \partial s_2}\cdot\right.$$

$$\left.\cdot\left(\frac{\partial w}{\partial s_2}-\frac{v}{R_2}\right)+\frac{1}{2}\left(\frac{\partial u}{\partial s_2}+\frac{\partial v}{\partial s_1}\right)\left(\frac{\partial^2 w}{\partial s_1^2}+\frac{\partial^2 w}{\partial s_2^2}\right)\right\}-\frac{1}{R_1}\left\{\frac{\partial v}{\partial s_1}-\frac{\partial u}{\partial s_2}\right.$$

$$-\left(\frac{\partial w}{\partial s_1}\right)\left(\frac{\partial w}{\partial s_2}\right)+2\left(\frac{\partial u}{\partial s_1}\right)\left(\frac{\partial u}{\partial s_2}\right)-\left(\frac{\partial u}{\partial s_1}\right)\left(\frac{\partial v}{\partial s_1}\right)+\left(\frac{\partial u}{\partial s_2}\right)\left(\frac{\partial v}{\partial s_2}\right)+\left(\frac{\partial u}{\partial s_2}\right)\left(\frac{w}{R_1}\right)+\left(\frac{\partial u}{\partial s_2}\right)$$

$$\left.\left(\frac{w}{R_2}\right)+\left(\frac{u}{R_1}\right)\left(\frac{\partial w}{\partial s_2}\right)-2\left(\frac{\partial v}{\partial s_1}\right)\left(\frac{w}{R_1}\right)\right\}-\frac{1}{R_2}\left\{\frac{\partial u}{\partial s_2}-\frac{\partial v}{\partial s_1}-\left(\frac{\partial w}{\partial s_1}\right)\left(\frac{\partial w}{\partial s_2}\right)+2\left(\frac{\partial v}{\partial s_1}\right)\left(\frac{\partial v}{\partial s_2}\right)\right.$$

$$-\left(\frac{\partial u}{\partial s_2}\right)\left(\frac{\partial v}{\partial s_2}\right)+\left(\frac{\partial u}{\partial s_1}\right)\left(\frac{\partial v}{\partial s_1}\right)+\left(\frac{\partial v}{\partial s_1}\right)\left(\frac{w}{R_2}\right)+\left(\frac{\partial v}{\partial s_1}\right)\left(\frac{w}{R_1}\right)+\left(\frac{v}{R_2}\right)\left(\frac{\partial w}{\partial s_1}\right)-$$

$$\left.-2\left(\frac{\partial u}{\partial s_2}\right)\left(\frac{w}{R_2}\right)\right\}$$

$$\lambda_{12} = \left\{\left(\frac{1}{R_2}-\frac{1}{R_1}\right)\frac{\partial v}{\partial s_1}-\frac{\partial^2 w}{\partial s_1 \partial s_2}\right\}\left(\frac{\partial^2 w}{\partial s_1^2}+\frac{w}{R_1^2}\right)+\left\{\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{\partial u}{\partial s_2}-\frac{\partial^2 w}{\partial s_1 \partial s_2}\right\}\cdot$$

$$\cdot\left(\frac{\partial^2 w}{\partial s_2^2}-\frac{w}{R_2^2}\right)$$

$$s_1 = R_1 \alpha_1 \quad , \quad s_2 = R_2 \alpha_2$$

ELEMENT ID: S-7

STRAIN DISPLACEMENT EQUATIONS:

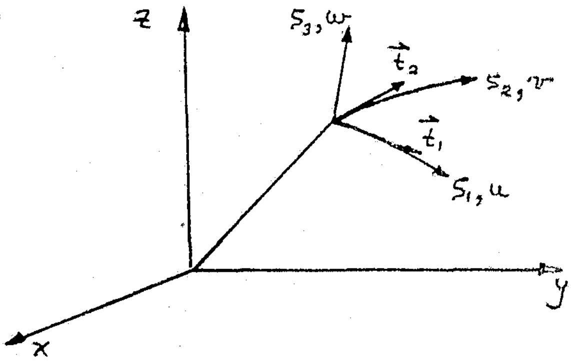
(Mushtari's Eqs.)

$$\{\varepsilon\} = \begin{Bmatrix} u_{,x} - z_{,xx}w + \frac{1}{2}w_{,x}^2 - \zeta_3 w_{,xx} \\ v_{,y} - z_{,yy}w + \frac{1}{2}w_{,y}^2 - \zeta_3 w_{,yy} \\ u_{,y} + v_{,x} - 2z_{,xy}w + w_{,x}w_{,y} - \zeta_2 w_{,xy} \end{Bmatrix}$$

DISCUSSION:

This element employs a linear stress-strain relation with all combinations of displacement and displacement gradients retained in the strain energy. A Lagrangian derivation is employed with a numerical solution obtained by a perturbation technique. The neutral equilibrium state is considered as is the solution through a critical point.

ELEMENT ID: S-7



ELEMENT TYPE:

Doubly curved shallow triangular shell

ASSUMED DISPLACEMENT SHAPE:

w = quintic polynomial in dimensionless triangular coordinates

u = cubic polynomial

v = cubic polynomial

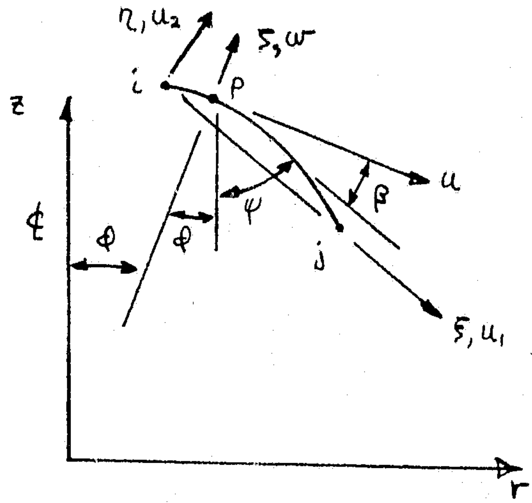
DESCRIPTION:

This element is an orthotropic doubly curved shell element of triangular shape with 3 corner nodes and 12 DOF ($u, u_{,x}, u_{,y}, v, v_{,x}, v_{,y}, w, w_{,x}, w_{,y}, w_{,xx}, w_{,yy}$). A centroidal node with 2 DOF (u,v) is eliminated by static condensation. The normal slope is constrained to a cubic variation along the edge. Membrane and bending deformation states are accounted for. Only geometric nonlinearities are considered.

REFERENCE: [62]

VARIATION OF THIS ELEMENT:

ADVANTAGES OR DISADVANTAGES:



ELEMENT ID: S-8

ELEMENT TYPE: Doubly curved Shell of Revolution

ASSUMED DISPLACEMENT SHAPE:

$$u_1 = H_{01}^{(1)}(\xi)u_{1i} + H_{02}^{(1)}(\xi)u_{1j} + H_{11}^{(1)}(\xi)u_{1,\xi_i} + H_{12}^{(1)}(\xi)u_{1,\xi_j}$$

$$u_2 = H_{01}^{(1)}(\xi)u_{2i} + H_{02}^{(1)}(\xi)u_{2j} + H_{11}^{(1)}(\xi)u_{2,\xi_i} + H_{12}^{(1)}(\xi)u_{2,\xi_j}$$

$$v = H_{01}^{(1)}(\xi)v_i + H_{02}^{(1)}(\xi)v_j + H_{11}^{(1)}(\xi)v_{,\xi_i} + H_{12}^{(1)}(\xi)v_{,\xi_j}$$

where $H_{01}^{(1)}(\xi)$, $H_{02}^{(1)}(\xi)$, $H_{11}^{(1)}(\xi)$, $H_{12}^{(1)}(\xi)$ are cubic Hermitian interpolation polynomials

Geometry: $\eta = (2\xi^3 - 3\xi^2 + 1)\eta_i + (3\xi^2 - 2\xi^3)\eta_j + (\xi^3 - 2\xi^2 + \xi)\tan\beta_i + (\xi^3 - \xi^2)\tan\beta_j$

DESCRIPTION:

This element is a doubly curved, orthotropic shell of revolution restricted to axisymmetric loads. It has too ring nodes and 6 DOF $(u, w, \chi, \varepsilon_s^{\circ}$, $v\ \gamma_{\varphi\theta}^{\circ})$ per node. Membrane and bending deformation states are included in the isoparametric derivation. Geometric and material nonlinearities are considered.

REFERENCE: [64]

VARIATION OF THIS ELEMENT: [67]

ADVANTAGES OR DISADVANTAGES:

ELEMENT ID: S-8

STRAIN DISPLACEMENT EQUATIONS: Sanders theory for small strains and moderate relations.

$$\epsilon_s = \epsilon_s^o + \frac{1}{2}(\phi_1^2 + \phi^2) + \zeta k_s$$

$$\epsilon_\theta = \epsilon_\theta^o + \frac{1}{2}(\phi_2^2 + \phi^2) + \zeta k_\theta$$

$$\gamma_{s\theta} = \gamma_{s\theta}^o + \phi_1 \phi_2 + 2 \zeta k_{s\theta}$$

$$\epsilon_s^o = \frac{du}{ds} + \frac{w}{R_1}, \quad \epsilon_\theta^o = \frac{1}{r}(u\cos\phi + w\sin\phi), \quad \gamma_{s\theta}^o = \frac{dv}{ds} - \frac{v\cos\phi}{r_o}, \quad k_s = -\frac{d\chi}{ds}, \quad k_\theta = -\frac{\cos\phi}{r}\chi$$

$$k_{s\theta} = \frac{1}{2r_o}\left\{\frac{1}{2}\frac{dv}{ds}\left(3\sin\phi - \frac{r_o}{R_1}\right) + \frac{v\cos\phi}{2R_1}\left(1 - \frac{3R_1\sin\phi}{r_o}\right)\right\}, \quad \chi = \frac{dw}{ds} - \frac{u}{R_1}, \quad \phi_1 = \frac{u}{R_1} - \frac{dw}{ds}$$

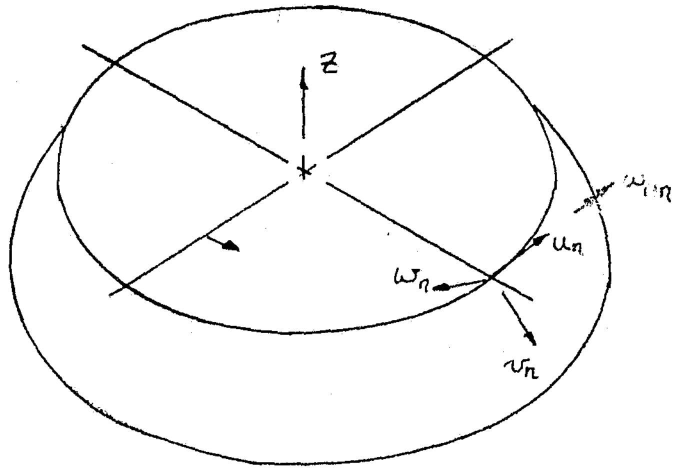
$$\phi_2 = \frac{v}{R_2}, \quad \phi = \frac{1}{2R_1 r_o}\left[(r_o v)_{,\phi}\right]$$

DISCUSSION:

This element employs a linear elastic stress-strain relation with only qudratic products of displacement gradients retained in the strain energy. The geometric stiffness matrix was developed to be used with an updated coordinate system approach. Integrations are performed numerically with up to eight order integration formulas available. Plastic material properties obey Hill's yield criterion. Up to 20 layers through the thickness may be used to calculate effective plastic forces and moments.

ELEMENT ID: S-9

ELEMENT TYPE: Axisymmetric Shell



ASSUMED DISPLACEMENT SHAPE:

A displacement shape is not assumed. The exact stiffness matrix, including nonlinear effects, is computed by using Fourier series expansions in the circumferential direction and a numerical integration procedure to solve the resulting differential equations.

DESCRIPTION:

Each element has two circumferential line nodes with four degrees of freedom per harmonic at each node, u_n, v_n, w_n and $\omega_{\theta n}$:

(Continued next pg.)

REFERENCE: [73], [74], [75], [76], [77], [78], [79].

VARIATION OF THIS ELEMENT:

An element in the form of a circumferential segment of a torus with meridional lines nodes is used in Ref. [80].

ADVANTAGES OR DISADVANTAGES:

The advantages of this formulation are many. First, the use of the numerical integration procedure, instead of an assumed displacement shape, yields "exact" stiffness matrices. Then, the transformation of the influence coefficient matrices into stiffness matrices, which is unique (other formulations using numerical integration perform all computations with the influence coefficients), allows the use of methods developed for finite element technology. These are: arbitrary branching, use of equilibrium corrections, use of Guyan reduction scheme and the method for solving eigenvalue problems.

ELEMENT ID: S-9

DESCRIPTION:

This program can analyze orthotropic thin shells of revolution, subjected to unsymmetric distributed loading or concentrated line loads, as well as thermal strains. Furthermore, a shell with arbitrary boundary conditions, under loads which vary arbitrarily with position and under a temperature variation through the thickness, is tractable with this program. The shell can consist of any combination of the following geometric shapes:

1. Ellipsoidal - spherical (offset from the axis of revolution allowed)
2. Ogival - toroidal
3. Modified ellipse shape
4. Conical - circular plate
5. Cylindrical
6. General point input geometry
7. Dummy geometry slot to be filled by the user
8. Discrete ring
9. Elastic support.

The shell wall cross-section can be a sheet, sandwich, or reinforced sheet or sandwich. The reinforcement can consist of rings and/or stringers, a waffle construction rotated at any angle to the principal coordinates, or an isogrid construction. General stiffness input options are also available. The reinforcement material properties can differ from those of the main shell, and a temperature variation can cause different properties in the two face sheets of a sandwich shell.

ELEMENT ID: S-9

DESCRIPTION:

The basic approach to the problem is to cut the structure into several shell regions. These regions need to be singly-connected shells, and can only have line loads applied at their end points. There are no restrictions on geometry, or uniform or thermal loads. The regions are further subdivided into several shell segments, each being free to have its own geometric shape, provided that the shape falls into one of the categories mentioned above.

There is a considerable latitude in what can be done within each shell segment. The thickness of any segment can be symmetrically tapered and it can contain up to 14 points of discontinuity, provided that the segment centerline remains continuous and describable by a single shell geometry. A temperature distribution through the thickness can be specified at three points in a homogeneous shell, and 4 points in a shell of rigid core sandwich construction. The distribution is considered to be linear between these points. Thus, it is possible to approximate temperature distributions other than linear distributions. In the event of physically discontinuous shell centerlines, a kinematic link is available for use in the analysis. The link relates displacements across the discontinuity. This link may be used between regions, and between segments within a region. Discrete offset rings are also available for use within or between regions.

ELEMENT ID: S-9

DISCUSSION:

This element is available in the STARS (Shell Theory Automated for Rotational Structures) computer programs developed at Grumman Aerospace Corporation and partially supported by NASA/MSFC. The following STARS programs are available:

1. STARS 2S (Static Analysis)
2. STARS 2V (Vibrations)
3. STARS 2B (Buckling)
4. STARS 2P (Geometric and Material Nonlinearities)

The use of an accurate shell theory to analyze structural shell problems usually involves complex mathematics and numerical techniques, which are nearly impossible to treat without the aid of automated procedures. On this basis, a digital computer program (STARS) based upon the Love-Reissner first order shell theory has been developed.

The shell equations are first cast (after Fourier expansion in the θ coordinate) in the form of eight coupled first-order equations of the form

$$[Y^{(n)}(y)],\ y = [D^{(n)}(y)][Y^{(n)}(y)] \qquad (1)$$

where φ, θ are the meridional and hoop coordinates, respectively, and n signifies the Fourier harmonic. These equations are then integrated by using some standard procedure such as the Runge-Kutta method. Eight influence coefficient solutions are thus obtained corresponding to eight independent initial condition vectors, and a particular solution for zero initial variables may be obtained for each independent loading pattern to be investigated. Due to the numerical difficulties caused by exponential growth or decay of the influence coefficient solutions, the shell must be segmented into pieces of limited size to obtain a satisfactory solution. Thus for each shell segment a set of (8+P) solutions is obtained, where P is the number of individual load patterns being considered. The total sets are then combined tc satisfy the segment continuity conditions, and the overall boundary conditions for the shell, thus yielding P solutions for stresses and displacements in the case of a static analysis.

These solutions represent influence coefficients for the shell segment which are then transformed into stiffness and load matrices. Finite element procedures used in the direct stiffness method are then applied to obtain a solution.

ELEMENT ID: S-9

DISCUSSION:

For the solution of buckling problems a new approach is taken. It is first recognized that the stability or vibrations problem for the shell is actually transcendental in the eigenvalue. Then, in successive passes through the program, the elastic stiffness matrix of the shell (K_0), and the prestressed shell stiffness matrix $[K_p(\lambda)]$, (which includes the nonlinear terms) for an assumed prestress value, are calculated from the differential equations. The two matrices are subtracted to isolate a matrix containing the effect of the prestress:

$$[K_I(\lambda)] = [K_p(\lambda)] - [K_0] \tag{2}$$

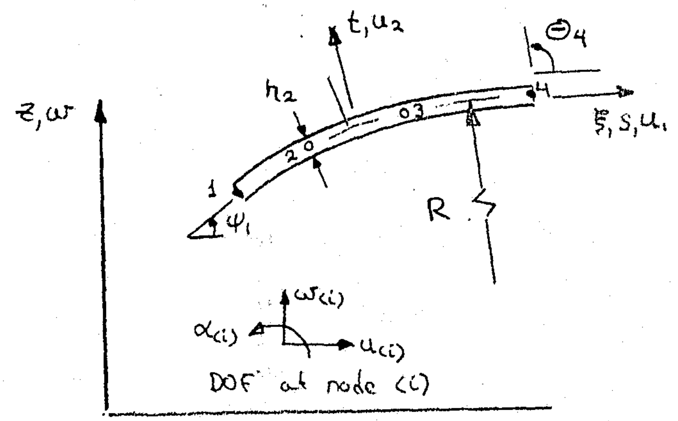
and a linear eigenvalue approximation is formulated:

$$([K_0] + \frac{\lambda_i}{\lambda_{i-1}} [K_I(\lambda i-1)])[\Delta] = [0] \tag{3}$$

where $[\Delta]$ is the deflection vector of the segment ends. As evident from the notation, Eq. (3) is solved by a series of iterations, with convergence being signaled by $\lambda_i/\lambda_{i-1} \to 1$. In this fashion the nonlinearity of the eigenvalue problem is retained. The iteration procedure has been found to exhibit quick convergence. A single iteration is equivalent to the process utilized in most finite element programs, whereas further passes include the refinements of accounting for the nonlinear dependence of the incremental stiffness matrix upon the eigenvalue. The formulation of Eq. (3) includes the effects of all 'consistent' nonlinear terms including predeformation.

An incremental Lagrangian approach is used for the geometric nonlinear analysis. The nonlinear shell formulation is based on a Love-Reissner-Kempner shell theory and is valid for small strains and moderate rotations.

ELEMENT ID: S-10



ELEMENT TYPE:

Doubly curved
shell of revolution

ASSUMED DISPLACEMENT SHAPE:

$$\begin{bmatrix} u \\ w \end{bmatrix} = \sum_{i=1}^{4} \phi_i(\xi) \begin{bmatrix} u_i \\ w_i \end{bmatrix} + \eta \sum_{i=1}^{4} \frac{1}{2} \phi_i(\xi) h_i \begin{bmatrix} -\sin\Theta_i \\ \cos\Theta_i \end{bmatrix} \alpha_i$$

and

$$\begin{bmatrix} r \\ z \end{bmatrix} = \sum_{i=1}^{4} \phi_i(\xi) \begin{bmatrix} r_i \\ z_i \end{bmatrix} + \eta \sum_{i=1}^{4} \frac{1}{2} \phi_i(\xi) h_i \begin{bmatrix} \cos\Theta_i \\ \sin\Theta_i \end{bmatrix}$$

$$\phi_{1,4}(\xi) = \frac{1}{16}(1 \pm \xi)(-1+9\xi^2), \quad \phi_{2,3}(\xi) = \frac{9}{16}(1 \mp 3\xi)(1-\xi^2)$$

DESCRIPTION:

This is a doubly curved, isotropic, shell of revolution. It has four nodes, two internal, with 3 DOF (u, w, α) per node. This is the degenerate isoparametric shell element of Ahmed, Ref. [83]. Membrane and bending stiffnesses are included in the derivation. The particular application includes geometric nonlinearities in a viscoelastic shell.

REFERENCE: [82]

VARIATION OF THIS ELEMENT: In Ref. [84] this element is used in a nonlinear elastic-viscoplastic analysis.

ADVANTAGES OR DISADVANTAGES:

ELEMENT ID: S-10

STRAIN DISPLACEMENT EQUATIONS:

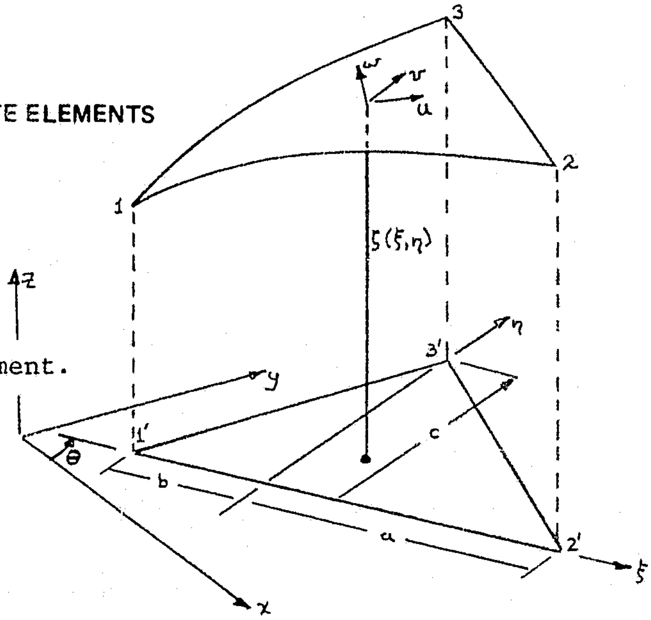
$$\begin{bmatrix} e_{ss} \\ e_{\theta\theta} \\ e_{st} \end{bmatrix}_{linear} = \begin{bmatrix} (1+\frac{\partial^1 u_1}{\partial s}) & 0 & 0 & \frac{\partial^1 u_2}{\partial s} \\ 0 & (1+\frac{^1u}{r}) & 0 & 0 \\ \frac{\partial^1 u_1}{\partial t} & 0 & (1+\frac{\partial^1 u_1}{\partial s}) & (1+\frac{\partial^1 u_2}{\partial t}) \end{bmatrix} \begin{bmatrix} \frac{\partial u_1}{\partial s} \\ \frac{u}{r} \\ \frac{\partial u_1}{\partial t} \\ \frac{\partial u_2}{\partial s} \end{bmatrix}$$

$$\begin{bmatrix} \eta_{ss} \\ \eta_{\theta\theta} \\ \eta_{st} \end{bmatrix}_{nl} = \begin{bmatrix} (\frac{\partial u_1}{\partial s})^2 + (\frac{\partial u_2}{\partial s})^2 \\ (\frac{u}{r})^2 \\ \frac{\partial u_1}{\partial s}\ \frac{\partial u_1}{\partial t} \end{bmatrix}$$

where the superscript 1 refers to the current configuration

DISCUSSION:

This element was developed by generalizing the infinitesimal theory of linear viscoelasticity to small strain large relation applications. The form given in this application is in terms of the Lagrangian strain rate and the second Piola-Kirchhoff stress tension. The viscoelastic material is characterized using a Preny series expansion for the relation modulus. A Lagrangian strain description is used with both conservative and nonconservative loads derived using the virtual work, The resulting equilibrium equations consist of the incremental and geometric stiffnesses.



ELEMENT ID: S-11

ELEMENT TYPE:

Triangular plate and shallow shell element.

ASSUMED DISPLACEMENT SHAPE:

$$u = a_1 + a_2\xi + a_3\eta + a_4\xi^2 + a_5\xi\eta + a_6\eta^2 + a_7\xi^3 + a_8\xi^2\eta + a_9\xi\eta^2 + a_{10}\eta^3$$

$$v = a_{11} + a_{12}\xi + a_{13}\eta + a_{14}\xi^2 + a_{15}\xi\eta + a_{16}\eta^2 + a_{17}\xi^3 + a_{18}\xi^2\eta + a_{19}\xi\eta^2 + a_{20}\eta^3$$

$$w = a_{21} + a_{22}\xi + a_{23}\eta + a_{24}\xi^2 + a_{25}\xi\eta + a_{26}\eta^2 + a_{27}\xi^3 + a_{28}\xi^2\eta + a_{29}\xi\eta^2 + a_{30}\eta^3$$
$$+ a_{31}\xi^4 + a_{32}\xi^3\eta + a_{33}\xi^2\eta^2 + a_{34}\xi\eta^3 + a_{35}\eta^4 + a_{36}\xi^5 + a_{37}\xi^3\eta^2 + a_{38}\xi^2\eta^3 + a_{39}\xi\eta^4 + a_{40}\eta^5$$

DESCRIPTION:

This element is an isotropic doubly curved, shallow, triangular shell element. It has three vertex nodes with 12 DOF ($u, v, w, u_\xi, v_\xi, w_\xi, u_\eta, v_\eta, w_\eta, w_{\xi\xi}, w_{\xi\eta}, w_{\eta\eta}$) per node and a controidal node with two DOF (u, v) that are removed by static condensation. It is developed in a cartesian coordinate system and includes both bending and stretching deformation states.

REFERENCE: [86]

VARIATION OF THIS ELEMENT:

ADVANTAGES OR DISADVANTAGES:

The element uses high precision displacement functions resulting in better accuracy with fewer elements than for elements with lower order displacement functions.

ELEMENT ID: S-11

STRAIN DISPLACEMENT EQUATIONS:

middle surface strains

$$\epsilon_{\xi\xi} = u_{\xi} - \zeta_{\xi\xi} w + \frac{1}{2} w_{\xi}^{2}$$

$$\epsilon_{\eta\eta} = v_{\eta} - \zeta_{\eta\eta} w + \frac{1}{2} w_{\eta}^{2}$$

$$\epsilon_{\xi\eta} = u_{\eta} + v_{\xi} - 2 \zeta_{\xi\eta} w + w_{\xi} w_{\eta}$$

bending strains

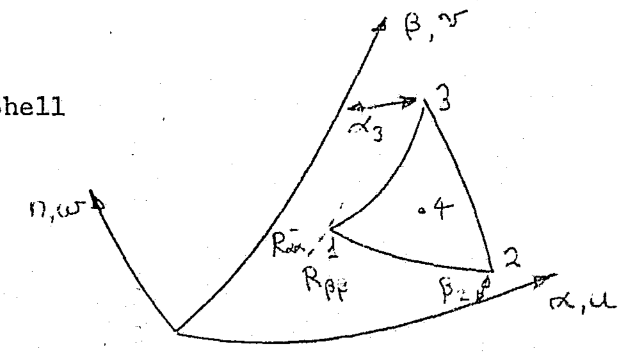
$$K_{\xi\xi} = -w_{\xi\xi}, \quad K_{\eta\eta} = -w_{\eta\eta}, \quad K_{\xi\eta} = -2w_{\xi\eta}$$

DISCUSSION:

This is the Cowper element of Ref. [87], extended to the large deflection regime. Linear elastic stress strain equations are used and all terms up to fourth order are retained in the strain energy. The formulation being based on the above strain-displacement equations is a Lagrangian formulation. The integration is performed exactly. The Newton-Rophson procedure is used to solve the equations of equilibrium.

ELEMENT ID: S-12

ELEMENT TYPE: Doubly curved Triangular Shell



ASSUMED DISPLACEMENT SHAPE: Cubic in area coordinates for membrane and bending displacements.

DESCRIPTION: Uses orthogonal curvilinear coordinate system to describe doubly curved geometry of element, with user-input coordinates and curvatures at the three corner nodes. Degrees of freedom are three displacements and their derivatives with respect to the curvilinear coordinates at the three corner nodes and the displacements at a central node, a total of 30. Constraints are imposed to obtain approximate slope continuity, using a Lagrange Multiplier Technique. Retains the Lagrange Multipliers explicitly in the set of unknowns to be solved.

REFERENCE: [119].

VARIATION OF THIS ELEMENT:

ADVANTAGES OR DISADVANTAGES: Advantages include simplicity due to cubic displacement forms, with competence due to using cubics for the membrane displacements.
Disadvantages include the need to handle Lagrange Multiplier and the fact that rigid body motion causes small element straining. The element shows accuracy in applications.

ELEMENT ID: S-12

STRAIN DISPLACEMENT EQUATIONS: Green's strain in a Lagrangian formulation is used. Effectively, the nonlinear terms correspond to the flat membrane equation

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2, \text{ etc.},$$

the membrane nonlinear term being dropped. Actually, a deep shell strain-displacement formulation with nonlinear terms as noted above is used.

DISCUSSION:

The desciption of the element is direct. However, it appears that the solution procedure is a stepwise linear incremental Lagrangian with a tangent stiffness (modified Newton-Raphson) iteration to avoid cumulative error. The total nonlinear strain must therefore be available, and the tangent stiffness appears to have retained all nonlinearities, ie., through the cubic terms in the total displacement. The geometric stiffness matrix does not explicitly appear in the equations.

Slope continuity is imposed at the midpoints of the sides of the element, using the Lagrange Multiplier procedure.

Instability is evaluated as the zero of the determinant of the tangent stiffness matrix.

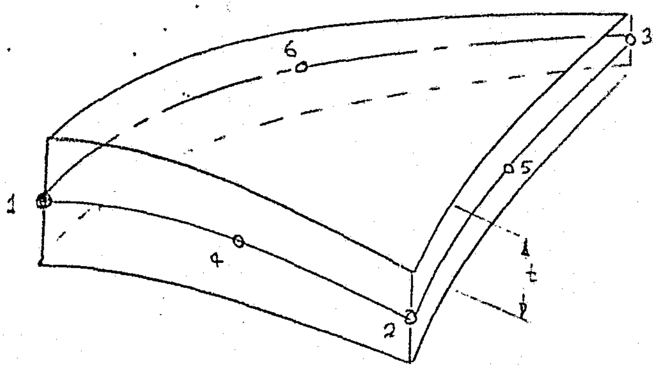
Numerical results are given for static analysis and for instability and nonlinear behavior cases, with good accuracy.

ELEMENT ID: S-13

ELEMENT TYPE: SHEBA6 (SHEBA3)
(General Shell Element

ASSUMED DISPLACEMENT SHAPE: Complete fifth order.

Figure:



DESCRIPTION: Number of nodes : 6
Degrees of freedom : Vertices

$u\ u,_x\ u,_y\ u,_{xx}\ u,_{xy}\ u,_{yy}$

$v\ v,_x\ v,_y\ v,_{xx}\ v,_{xy}\ v,_{yy}$

$w\ w,_x\ w,_y\ w,_{xx}\ w,_{xy}\ w,_{yy}$

Midside nodes, u_n, v_n, w_n, n stands for a local direction normal to the edge, all together 63.

REFERENCE: [100]

Isotropic or Anisotropic
Deformation type: stretching, in-plane, shearing, bending without shear

VARIATION OF THIS ELEMENT:

Eliminations of mid-side-nodes leads to SHEBA3.

ADVANTAGES OR DISADVANTAGES:

Advantages: Unachieved accuracy even in case of very coarse grids. Successful handling of the most complicated shell problems. Full compatibility.

Disadvantages: Application needs experienced staff. Grid lines in parameter plane must be straight. Jumps in thickness provide difficulties.

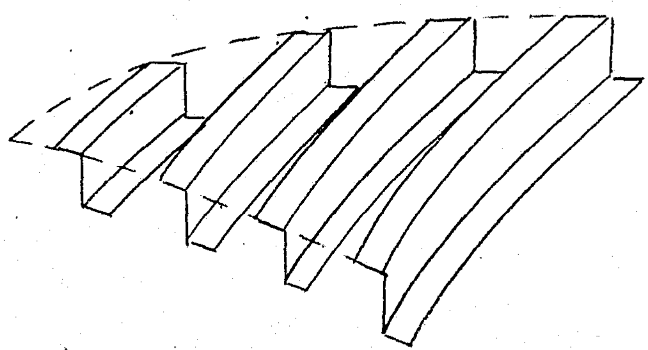
Discussion: Based on the concept of natural stress invariants for large rotations.

ELEMENT ID: S-14

ELEMENT TYPE: ARCUS 6 (ARCUS 3)
(Stiffener layer fully compatible with SHEBA6 (SHEBA 3))

ASSUMED DISPLACEMENT SHAPE: Fifth order

Figure:



DESCRIPTION:

Number of nodes)	see SHEBA 6 (SHEBA 3)
Degrees of freedoms)	
Deformation type :	stretching, bending without shear, torsion (St. Venant + Wagner)

REFERENCE: [101]

VARIATION OF THIS ELEMENT: Free beam is under development (applicable to curved grids)

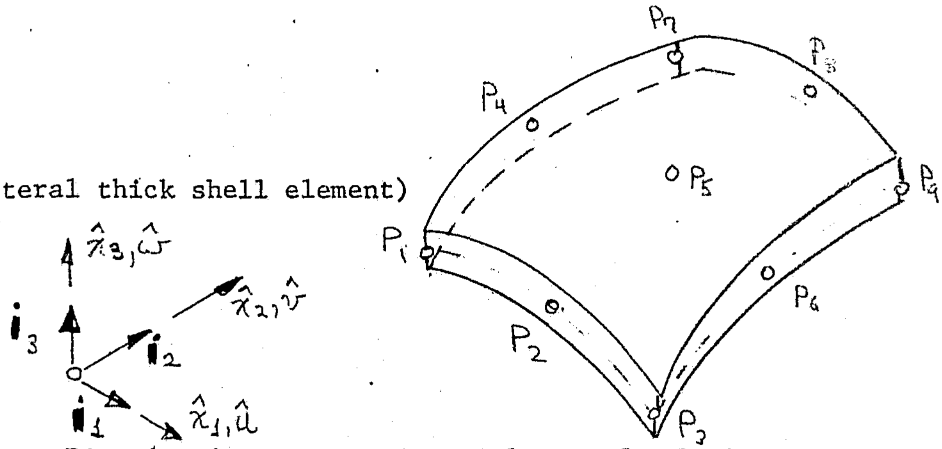
ADVANTAGES OR DISADVANTAGES:

Discussion: Based on the concept of natural stress invariants for large rotations.

ELEMENT ID: S-15

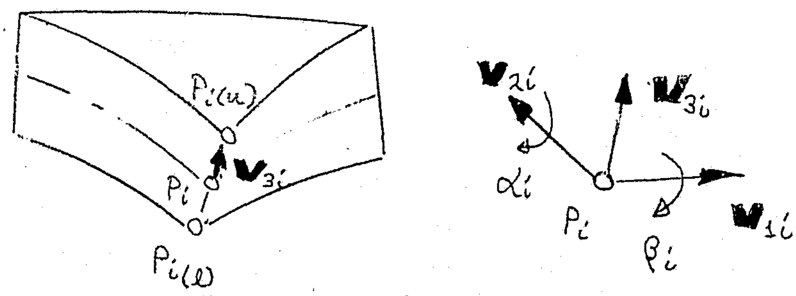
ELEMENT TYPE: QUABC9

(Quadrilateral thick shell element)



ASSUMED DISPLACEMENT SHAPE: Biquadratic Lagrance interpolation for both displacements and rotations.

Figure:



DESCRIPTION:

Derived from 3D continuum model by "degenerating" assumption in thickness direction.

Number of nodes	:	9
Degrees of freedom		45 (per node, i.e., u, v, w, x, β)
Deformation type	·	stretching and in-plane shear; bending behavior stems solely from transverse shear.

REFERENCE: [102]

VARIATION OF THIS ELEMENT: Possible extension to higher order interpolation scheme.

ADVANTAGES OR DISADVANTAGES:

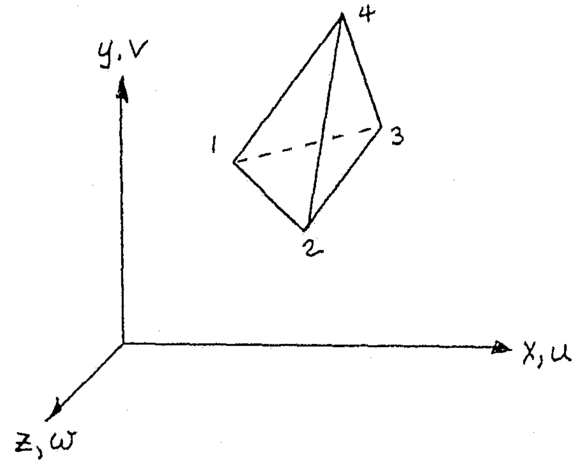
Advantages: Good engineering accuracy, simple representation of geometry.

Disadvantages: Restriction to analysis of "thick" shells, due to doubtful validity of "reduced integration technique" in case of "thin" shells.

Discussion: Performance has been investigated within internal research, therefore the nonlinear portion of QUABC9 is not incorporated in ASKA.

ELEMENT ID: C-1

ELEMENT TYPE: Tetrahedron



ASSUMED DISPLACEMENT SHAPE:

$$u = ax + \phi_1 y + (f - \phi_7) z + \phi_9$$

$$v = (d - \phi_1) x + by + \phi_4 z + \phi_5$$

$$w = \phi_7 x + (e - \phi_4) y + cz + \phi_8$$

DESCRIPTION:

This element is an isotropic tetrahedron with 4 nodes and 3 DOF (u, v, w) per node. It accounts for material nonlinearities, but does not consider geometric nonlinearities. A three-dimensional state of constant stress is obtained.

REFERENCE: [19]

VARIATION OF THIS ELEMENT:

ADVANTAGES OR DISADVANTAGES:

ELEMENT ID: C-1

STRAIN DISPLACEMENT EQUATIONS:

$$\varepsilon_x = \varepsilon_{xe} + \alpha T + \varepsilon_{xpt} \qquad \gamma_{xy} = \gamma_{xye} + \gamma_{xypt}$$

$$\varepsilon_y = \varepsilon_{ye} + \alpha T + \varepsilon_{ypt} \qquad \gamma_{yz} = \gamma_{yze} + \gamma_{yzpt}$$

$$\varepsilon_z = \varepsilon_{ze} + \alpha T + \varepsilon_{zpt} \qquad \gamma_{zx} = \gamma_{zxe} + \gamma_{zxpt}$$

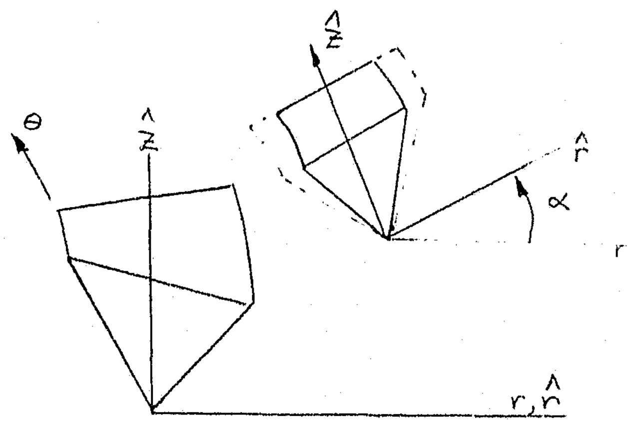
e - elastic
pt - plastic

DISCUSSION:

This derivation considers material nonlinearities for an isotropic tetrahedron. Plastic strains are converted to a column matrix of plastic loads.

ELEMENT ID: C-2

ELEMENT TYPE: Triangular, Toroidal Continuum Element



ASSUMED DISPLACEMENT SHAPE:

$$u_r = a_1 \hat{r} + a_2 \hat{z}$$

$$u_z = a_3 \hat{r} + a_4 \hat{z}$$

DESCRIPTION:

This element is an isotropic toroidal ring element developed for a large displacement, moderate rotation, elastic-plastic dynamic problem. It has 3 ring nodes and 2 DOF (u_r, u_z) per node. Convected coordinates are used to derive the elemental matrices. A two dimensional deformation state is assumed. Both geometric and material nonlinearities are included.

REFERENCE: [54]

VARIATION OF THIS ELEMENT:

ADVANTAGES OR DISADVANTAGES:

The method is stated as being computationally efficient.

ELEMENT ID: C-2

STRAIN DISPLACEMENT EQUATIONS:

$$\begin{Bmatrix} \hat{\epsilon}_r \\ \hat{\epsilon}_z \\ 2\hat{\epsilon}_{rz} \end{Bmatrix} = \begin{bmatrix} \partial/\partial\hat{r} & 0 \\ 0 & \partial/\partial\hat{z} \\ \partial/\partial\hat{z} & \partial/\partial\hat{r} \end{bmatrix} \begin{Bmatrix} \hat{u}_r^{\,def.} \\ \hat{u}_z^{\,def} \end{Bmatrix}$$

$\epsilon_\theta = u_r/r$ is assumed to be linear.

DISCUSSION:

This element is stated as being able to account for material nonlinearities. Because convected coordinates are used, the geometric nonlinearities that arise are accounted for by transformations between the convected and global systems. The strains are linearly related to deformation displacements relative to the convected coordinate system. The dynamic equations of equilibrium are solved by a central difference explicit integration procedure which uses total nodal forces.

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